Stability and normal forms (around leaves in Poisson geometry)

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Based on joint work with R.L. Fernandes.

Main motivation: Conn's linearization theorem.

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Actions of Lie groups

Lie group actions: the orbit data

Let G be a Lie group acting on a manifold M and let $\mathcal{O}$ be the orbit through a point x. Consider:

- the isotropy group at x, call it G_x.
- the normal space at x to the orbit, call it V.
- the induced (linear) action of G_x on V (isotropy action).

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Local form: the slice theorem

Local model out of the orbit data: the G-manifold

$$G \times_{G_x} V = (G \times V)/G_x .$$

Theorem (the slice theorem)

If G is compact, then $\mathcal{O}$ admits a G-invariant neighborhood $\mathcal{U}$, G-diffeomorphic to $G \times_{G_x} V$.

Read: ($\mathcal{O}$ and G_x compact) instead of (G-compact).

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Theorem (stability theorem, Hirsch 1980, Stowe 1983)

If $\mathcal{O}$ is compact and $H^1(G_x; V) = 0$, then $\mathcal{O}$ is stable, i.e. any other action of G on M which is close enough to the original one has at least one orbit diffeomorphich to $\mathcal{O}$.

Remark: one can also say how many leaves are diff. to $\mathcal{O}$ (at least as many as $H^0(G_x, V)$).

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II. FOLIATIONS

Foliations: the leaf data

Let $\mathcal{F}$ be a foliation on a manifold M and let L be a leaf of the foliation through a point x. Consider

- the fundamental group $\Gamma_x = \pi(L, x)$.
- the normal space at x to the leaf L, call it V.
- the (linear) holonomy action of Γ_x on $\nu_x(L)$.

Also consider the holonomy group at x:

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Local form: Local Reeb stability

Local form out of the local data: the foliated manifold

$$\tilde{L} \times_{\Gamma_x} V = (\tilde{L} \times V)/\Gamma_x$$

with the foliation coming from the product foliation (with leaves $\tilde{L} \times \{v\}$).

Theorem (local Reeb stability)

If L is a compact leaf and Γ_x' is finite, then L admits a saturated neighborhood $\mathcal{U}$ which is diffeomorphic to $\tilde{L} \times_{\Gamma_x} V$ as foliated manifolds.

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If L is compact and $H^1(\Gamma_x; V) = 0$, then L is stable i.e. any other foliation on M which is close enough to the original one, has at least one leaf which is diffeomorphic to L.

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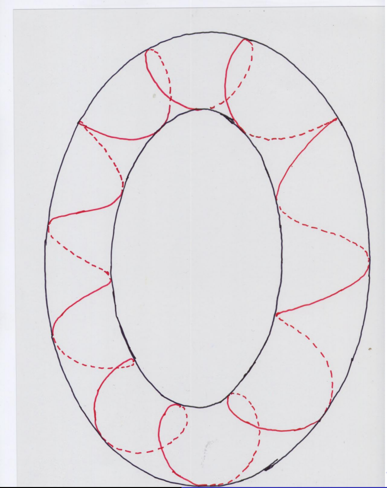
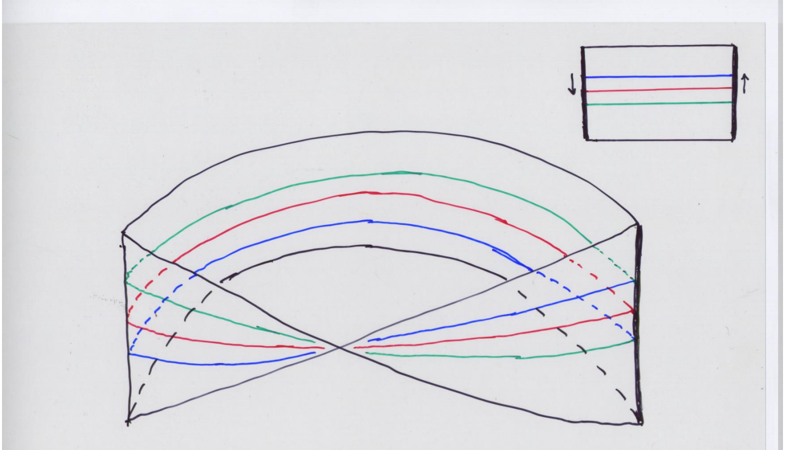
On the torus

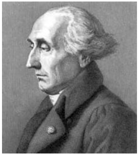


On the Moebius band



III: POISSON GEOMETRY: INTRODUCTION

Origins


J.L. Lagrange 1736-1813

1809: Lagranian mechanics


S. Poisson 1781-1840

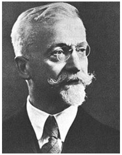
1809: Poisson brackets


W.R. Hamilton 1805-1865

1833: Hamiltonian mechanics

In 1831-1837, A.L. Cauchy: clear present. of Lagrange's methods, using Hamilton's formalism and Poisson brackets.


Marius Sophus Lie 1842-1899
Lie (pseudo-) groups
Linear Poisson structures


Ellie Joseph Cartan 1869-1951
Lie theory, EDS, etc

$\overset{Weil, Chern, etc}{\Longrightarrow}$ modern geometry, Lie theory, geometry of PDE's

Various descriptions

A Poisson structure on M is a Lie bracket $\{-,-\}$ on $C^\infty(M)$ s.t.

$$\{f,-\}:C^\infty(M)\longrightarrow C^\infty(M)$$

acts as a vector field (the Hamiltonian vector field X_f).

Equivalently, it is a bivector π on M with the property that

$$\{f,g\}:=\pi(df,dg)$$

satisfies Jacoby (or $[\pi,\pi]=0$).

Locally: determined by fcts $\pi_{i,j}=\{x_i,x_j\}$; $\pi=\sum \pi_{i,j}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$.

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- S/G where S is symplectic and G acts on S by symplectomorphisms.

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Conn's theorem

Let (M, π)-Poisson, $x \in M$ singular point (i.e. $\pi_x = 0$).

Local data: $\mathfrak{g}_x$- the isotropy Lie algebra at x. Locally:

$$c_{i,j}^k = \frac{\partial \pi_{i,j}}{\partial x_k}(x).$$

Local model out of the local data: $\mathfrak{g}_x^*$.

Theorem (J. Conn, 1980)

If $\mathfrak{g}_x$ is semi-simple of compact type, then x admits a neighborhood $\mathcal{U}$ which is Poisson diffeomorphic to $\mathfrak{g}_x^*$.

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The symplectic leaves

In a Poisson manifold (M, π): the hamiltonians X_f define an integrable distribution on M. A symplectic leaf is a maximal integral submanifold.

Equivalently: $x, y \in M$ are in the same leaf iff there exists a piecewise smooth curve joining x to y made of integral curves of hamiltonian vector fields:

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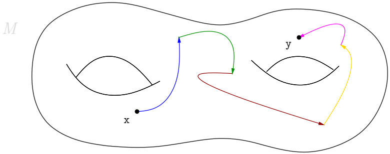

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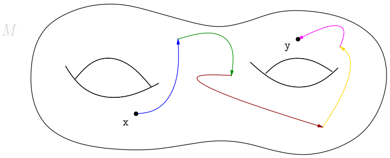

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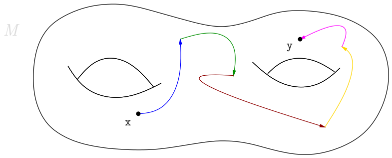

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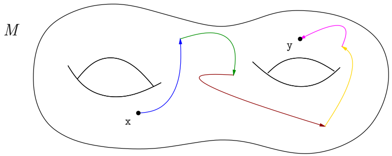

The symplectic leaves

$\Longrightarrow$ a partition of M into symplectic submanifolds.

Actually: a Poisson manifold $=$ a manifold with a (possibly singular) symplectic foliation.

Basic examples:

- $\mathfrak{g}^* \Longrightarrow$ coadjoint orbits.
- S/G, with S a hamiltonian G-space $\Longrightarrow$ the symplectic reductions.

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Ordinary Geometry	Poisson Geometry
Tangent bundle TM with usual Lie bracket.	Cotangent bundle T^*M: related to TM by $\pi^\sharp : T^*M \longrightarrow TM$, $df \mapsto X_f$, bracket $[df, dg] = d\{f, g\}$.
Curves: $\gamma : I \longrightarrow M$ (with derivatives $\dot{\gamma} : I \longrightarrow TM$).	Cotangent curves: $\gamma : I \longrightarrow M$ plus "cotangent derivatives" $a : I \longrightarrow T^*M$ which are $\pi^\sharp$-related to $\dot{\gamma}$.
Homotopy of curves.	Cotangent homotopies.
k-forms: $\Omega^k(M) = \Gamma(\wedge^k T^*M)$.	Multivector fields: $\mathfrak{X}^k(M) = \Gamma(\wedge^k TM)$;
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Tangent bundle TM with usual Lie bracket.	Cotangent bundle T^*M: related to TM by $\pi^\sharp : T^*M \longrightarrow TM$, $df \mapsto X_f$, bracket $[df, dg] = d\{f, g\}$.
Curves: $\gamma : I \longrightarrow M$ (with derivatives $\dot{\gamma} : I \longrightarrow TM$).	Cotangent curves: $\gamma : I \longrightarrow M$ plus "cotangent derivatives" $a : I \longrightarrow T^*M$ which are $\pi^\sharp$-related to $\dot{\gamma}$.
Homotopy of curves.	Cotangent homotopies.
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IV. POISSON GEOMETRY: LOCAL DATA AND THE LOCAL MODEL

Poisson geometry: the local data and the local model

Start with (M, π) Poisson manifold, $x \in M$.

The symplectic leaf through x

Consider the symplectic leaf through x.

$$S_x = \{\gamma(1) : \gamma - \text{cotangent path starting at } x\}.$$

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Isotropy at x

The isotropy at x, or the Poisson fundamental group at x:

$$G_x := \{\text{cotangent loops at } x\}/\{\text{cotangent homotopy}\}.$$

If Hausdorff, it is a Lie group (this is mostly the case, and we understand when this fails).

Infinitesimally,

$$\mathfrak{g}_x = \nu_x^* \subset T^*M$$

(with the bracket coming from the one on 1-forms).

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Poisson universal cover at x

The universal cover at x:

$$P_x := \{\text{cotangent paths at } x\}/\text{cotangent homotopy}.$$

It is a (smooth) principal G_x-bundle over S_x precisely when G_x is smooth.

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- a symplectic manifold S.
- a Lie group G.
- a principal G-bundle over S.

$\Longrightarrow$ a Poisson manifold

$$P[\mathfrak{g}^*] = P \times_G \mathfrak{g}^* = (P \times \mathfrak{g}^*)/G.$$

(the coadjoint bundle of the principal bundle).

Here,

$$P \times \mathfrak{g}^*$$

carries a Poisson structure which combines the symplectic structure on S and the linear Poisson structure on $\mathfrak{g}^*$. (it uses a connection!).

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V. POISSON GEOMETRY: NORMAL FORMS AND STABILITY

Normal form

Theorem (together with I. Marcut, based on joint work with L.R. Fernandes and discussions with D.M. Torres)

If S_x is compact, G_x is compact and P_x is 2-connected, then S_x admits a neighborhood $\mathcal{U}$ which is Poisson diffeomorphic to $P_x \times_{G_x} \mathfrak{g}_x^*$.

Example: coadjoint orbits for semi-simple Lie algebras of compact type.

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Example: coadjoint orbits for semi-simple Lie algebras of compact type.

Stability

Theorem (with R.L. Fernandes)

If S_x is compact and $H^2_\pi(M, S_x) = 0$ then S is stable.

Here:

- $H^2_\pi(M, S_x)$ is defined similar to Poisson cohomology, by "restricting to S_x. It is finite dimensional and computable!
- If G_x is compact, then

$$H^2_\pi(M, S_x) \hookrightarrow H^1(G_x; \nu_x).$$

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About the proof of Conn's theorem

About the proof of the stability theorem

It is a combination of several ingredients:

- Geometric: a coupling construction, like in symplectic geometry.
- Algebraic: the structure that governs complicated equations.
- Analytic: the actual proof, based on minimizing a certain functional defined on the Sobolev space of sections and taking advantage of the ellipticity of the operators (complexes) involved.

Main idea: work in a tubular neighborhood and look for leaves

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