# Connections and Dirac operators with torsion


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Relations between different objects on a Riemannian manifold $(M^n, g)$:

$\boxed{\text{geometric structures}}$

curvature, contact str.,
almost complex str. . .

$\mathrm{Ric} = \lambda \cdot g$

adapted connection,
Berger's thm if $\nabla = \nabla^g$

$\boxed{\text{solutions of field eqs.}}$ $\longleftrightarrow$ $\boxed{\text{new invariant connection } \nabla \text{ and its holonomy}}$

Einstein eq., twistor eq.,
Killing eq., parallel tensor / spinor. . .

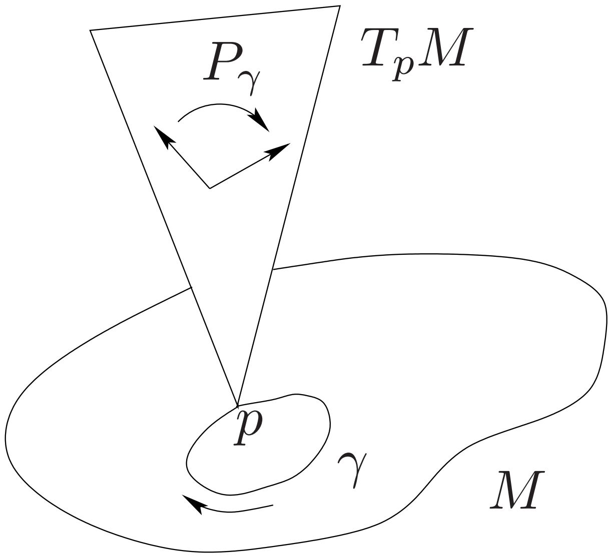
$\mathrm{Hol}(M; \nabla) \subset \mathrm{SO}(n)$

- Henceforth: $\nabla^g$ = Levi-Civita connection

## Holonomy group of a connection $\nabla$

- $\gamma$: closed path through $p \in M$, $P_\gamma : T_pM \to T_pM$ parallel transport
- $P_\gamma$ isometry $:\Leftrightarrow$: $\nabla$ metric
- $C_0(p)$: null-homotopic $\gamma$'s

$$\mathrm{Hol}_0(M;\nabla) := \{P_\gamma \mid \gamma \in C_0(p)\} \subset \mathrm{SO}(n)$$




**Thm (Berger / Simons, $\geq 1955$).** The reduced holonomy $\mathrm{Hol}_0(M;\nabla^g)$ of the LC connection $\nabla^g$ is either that of a symmetric space or

$$\mathrm{Sp}(n)\mathrm{Sp}(1)\ [\mathrm{qK}],\ \mathrm{U}(n)\ [\mathrm{K}],\ \underbrace{\mathrm{SU}(n)\ [\mathrm{CY}],\ \mathrm{Sp}(n)\ [\mathrm{hK}],\ G_2,\ \mathrm{Spin}(7)}_{\mathrm{Ric}=0}.$$

All of them admit a $\nabla^g$-parallel object and will be called

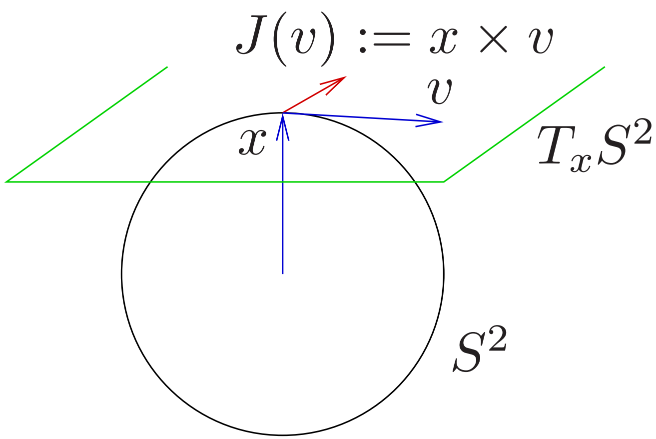
***'integrable geometries'***

## Examples of non-integrable geometries

### Example 1: almost Hermitian mnfd

- $(S^6, g_{\mathrm{can}})$: $S^6 \subset \mathbb{R}^7$ has an almost complex structure $J$ $(J^2 = -\mathrm{id})$ inherited from "cross product" on $\mathbb{R}^7$.
- $J$ is not integrable, $\nabla^g J \neq 0$
- **Problem (Hopf):** Does $S^6$ admit an (integrable) complex structure ?

for $S^2 \subset \mathbb{R}^3$:




$J$ is an example of a nearly Kähler structure: $\nabla^g_X J(X) = 0$

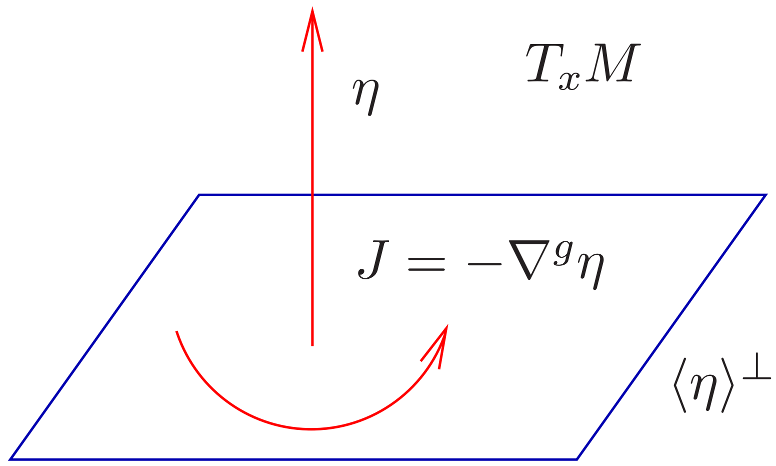
**More generally:** $(M^{2n}, g, J)$ almost Hermitian mnfd: $J$ almost complex structure, $g$ a compatible Riemannian metric.

**Fact:** structure group $G \subset \mathrm{U}(n) \subset \mathrm{SO}(2n)$, but $\mathrm{Hol}_0(\nabla^g) = \mathrm{SO}(2n)$.

Examples: twistor spaces $(\mathbb{CP}^3, F_{1,2})$ with their nK str., $\mathrm{SL}(2,\mathbb{C})_{\mathbb{R}}$, compact complex mnfd with $b_1(M)$ odd ($\nexists$ Kähler metric) . . .

## Example 2 – contact mnfd

- $(M^{2n+1}, g, \eta)$ contact mnfd,
$\eta$: 1-form ($\cong$ vector field)
- $\langle\eta\rangle^\perp$ admits an almost complex structure $J$ compatible with $g$




- Contact condition: $\eta \wedge (d\eta)^n \neq 0 \Rightarrow \nabla^g\eta \neq 0$, i. e. contact structures are never integrable ! (no analogue on Berger's list)

- structure group: $G \subset \mathrm{U}(n) \subset \mathrm{SO}(2n+1)$

Examples: $S^{2n+1} = \frac{\mathrm{SU}(n+1)}{\mathrm{SU}(n)}$, $V_{4,2} = \frac{\mathrm{SO}(4)}{\mathrm{SO}(2)}$, $M^{11} = \frac{G_2}{\mathrm{Sp}(1)}$, $M^{31} = \frac{F_4}{\mathrm{Sp}(3)}$

## Example 3 – Mnfds with $G_2$- or $\mathrm{Spin}(7)$-structure $(\dim = 7, 8)$

- $G_2$ has a 7-dimensional irred. representation,
- $\mathrm{Spin}(7)$ has a spin representation of dimension $2^3 = 8$.

Examples: $S^7 = \frac{\mathrm{Spin}(7)}{G_2}$, $M^{AW}_{k,l} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,l}}$, $V_{5,2} = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}$, $M^8 = \frac{G_2}{\mathrm{SO}(4)} \ldots$

## Example 4 – 5-dim. $\mathrm{SO}(3)$-mnfd   [Bobienski-Nurowski, 2007]

- modelled on the geometry of the symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3)$

$\exists$ two nonequivalent embeddings $\mathrm{SO}(3) \to \mathrm{SO}(5)$:

* as upper diagonal block matrices:'$\mathrm{SO}(3)_{\mathrm{st}}$'
* by the irreducible 5-dim. representation of $\mathrm{SO}(3)$: '$\mathrm{SO}(3)_{\mathrm{ir}}$'

**Fact:** $\mathrm{SO}(3)_{\mathrm{ir}}$ is the isotropy group of a symmetric $(3,0)$-tensor on $\mathbb{R}^5$ that is deeply related to Cartan's isoparametric hypersufaces in spheres

**Dfn.** A 5-manifold with a $\mathrm{SO}(3)_{\mathrm{ir}}$-structure is a manifold with a reduction of the frame bundle to $\mathrm{SO}(3)_{\mathrm{ir}}$.

Examples: $\mathrm{SO}(4)/\mathrm{SO}(2)$, solvable Lie groups [Chiossi-Fino, 2008], topological constructions, but not $S^5$, $\mathbb{RP}^5$...

**Thm.** If $M^5$ admits a $\mathrm{SO}(3)_{\mathrm{ir}}$-structure, then $p_1(M^5) \in H^4(M^5;\mathbb{Z})$ is divisible by $5$, $w_1(M^5) = 0$, $w_4(M^5) = 0$, $w_5(M^5) = 0$.

[IA-Friedrich, 2009]

**N.B.** Non-integrable geometries are not necessarily homogeneous. Some of those who *are* homogeneous fall into the following class:

**Example 5 – naturally reductive homogeneous space**

$M = G/H$ reductive space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\langle , \rangle$ a scalar product on $\mathfrak{m}$.

The PFB $G \to G/H$ induces a metric connection $\nabla$ with torsion

$$T(X, Y, Z) := -\langle [X, Y]_{\mathfrak{m}}, Z \rangle.$$

**Dfn.** $M = G/H$ is called *naturally reductive* if $T \in \Lambda^3(M)$

Naturally reductive spaces have the properties $\nabla T = \nabla \mathcal{R} = 0$
$\longrightarrow$ direct generalisation of symmetric spaces

Special geometries $\cong$ mnfds with geometric structures that are not defined through $\nabla^g$-parallel objects

## General philosophy:

Given a mnfd $M^n$ with $G$-structure ($G \subset \mathrm{SO}(n)$), replace $\nabla^g$ by a *metric connection $\nabla$ with torsion that preserves the geometric structure!*

$$\text{torsion:} \quad T(X,Y,Z) \ := \ g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as $\nabla^g$)

$$\Rightarrow \quad g(\nabla_X Y, Z) \ = \ g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

1) representation theory yields

- a clear answer *which* $G$-structures admit such a connection; if existent, it's unique and called the *'characteristic connection'*

- a *classification scheme* for $G$-structures with characteristic connection: $T_x \in \Lambda^3(T_x M) \stackrel{G}{=} V_1 \oplus \ldots \oplus V_p$

2) Analytic tool: Dirac operator $\not{D}$ of the metric connection with torsion $T/3$: *'characteristic Dirac operator'* (generalizes the Dolbeault operator)

## Difficulties:

(1) $\mathrm{Hol}_0(M;\nabla)$ needs not to be closed inside $\mathrm{SO}(n)$!

(2) The holonomy representation on $TM$ needs not to be irreducible for irreducible manifolds! (see contact case)

$\longrightarrow$ Larger variety of holonomy groups possible, but

- classification impossible: no ‘Berger Theorem’
- no ‘de Rham splitting Theorem’

**Thm (Holonomy Principle).** If a metric connection $\nabla$ admits a parallel spinor / tensor $\alpha$ ($\nabla\alpha = 0$), its holonomy group is contained in the isotropy group of the parallel object,

$$\mathrm{Hol}_0(\nabla) \subset \mathrm{Iso}(\alpha) \ := \ \{A \in \mathrm{SO}(n) \,|\, A^*\alpha = \alpha\}.$$

For (almost) all interesting objects the isotropy groups are known.

## The characteristic connection of a geometric structure

Fix $G \subset \mathrm{SO}(n)$, $\Lambda^2(\mathbb{R}^n) \cong \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, $\mathcal{F}(M^n)$: frame bundle of $(M^n, g)$.

**Dfn.** A geometric $G$-structure on $M^n$ is a $G$-PFB $\mathcal{R}$ which is subbundle of $\mathcal{F}(M^n)$: $\mathcal{R} \subset \mathcal{F}(M^n)$.

Choose a $G$-adapted local ONF $e_1, \ldots, e_n$ in $\mathcal{R}$ and define *connection 1-forms of* $\nabla^g$*:*

$$\omega_{ij}(X) := g(\nabla^g_X e_i, e_j), \quad g(e_i, e_j) = \delta_{ij} \ \Rightarrow \ \omega_{ij} + \omega_{ji} = 0.$$

Define a skew symmetric matrix $\Omega$ with values in $\Lambda^1(\mathbb{R}^n) \cong \mathbb{R}^n$ by $\Omega(X) := \big(\omega_{ij}(X)\big) \in \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ und set

$$\Gamma := \mathrm{pr}_{\mathfrak{m}}(\Omega).$$

- $\Gamma$ is a 1-Form on $M^n$ with values in $\mathfrak{m}$, $\Gamma_x \in \mathbb{R}^n \otimes \mathfrak{m}$ $(x \in M^n)$ [“intrinsic torsion”, Swann/Salamon]

Fact: $\Gamma = 0 \Leftrightarrow \nabla^g$ is a $G$-connection $\Leftrightarrow \mathrm{Hol}(\nabla^g) \subset G$

Via $\Gamma$, geometric $G$-structures $\mathcal{R} \subset \mathcal{F}(M^n)$ correspond to irreducible components of the $G$-representation $\mathbb{R}^n \otimes \mathfrak{m}$.

**Thm.** A geometric $G$-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ admits a metric $G$-connection with antisymmetric torsion iff $\Gamma$ lies in the image of $\Theta$,

$$\Theta : \Lambda^3(M^n) \to T^*(M^n) \otimes \mathfrak{m}, \quad \Theta(T) \ := \ \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner T)\,.$$

If such a connection exists, it is called the *characteristic connection* $\nabla^c$ and it is unique in all known cases; its torsion is essentially $\Gamma$ and $\mathrm{Hol}(\nabla^c) \subset G$.

If existent, we can thus replace the (unadapted) LC connection by some new unique metric $G$-connection!

## Some characteristic connections

**Example 1 – almost Hermitian mnfd** [Friedrich, Ivanov 2000]

$\exists$ a char. connection $\nabla$ $\Leftrightarrow$ Nijenhuis tensor $g(N(X,Y),Z) \in \Lambda^3(M)$,

$$g(\nabla_X Y, Z) \;:=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2}\,[g(N(X,Y),Z) + d\Omega(JX,JY,JZ)]$$

- $\mathrm{Hol}_0(\nabla) \subset \mathrm{U}(n) \subset \mathrm{SO}(2n)$
- In the nearly-Kähler case it is the *Gray connection* and satisfies $\nabla T = 0$ [Kirichenko, 1977]

**Example 2 – contact mnfd** [Friedrich, Ivanov 2000]

A large class admits a char. connection $\nabla$, and $\mathrm{Hol}_0(\nabla) \subset \mathrm{U}(n) \subset \mathrm{SO}(2n+1)$. For Sasaki manifolds, the formula is particularly simple,

$$g(\nabla_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2}\eta \wedge d\eta(X,Y,Z),$$

and $\nabla T = 0$ holds. [Kowalski-Wegrzynowski, 1987 for Sasaki]

## Example: $G_2$ structures in dimension 7

Fix $G_2 \subset \mathrm{SO}(7)$, $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}^7 \cong \mathfrak{g}_2 \oplus \mathbb{R}^7$.

Intrinsic torsion $\Gamma$ lies in $\mathbb{R}^7 \otimes \mathfrak{m}^7 \cong \mathbb{R}^1 \oplus \mathfrak{g}_2 \oplus \mathrm{S}_0(\mathbb{R}^7) \oplus \mathbb{R}^7 =: \bigoplus_{i=1}^4 W_i$

$\Rightarrow$ four classes of geometric $G_2$ structures [Fernandez-Gray, '82]

- Decomposition of 3-forms: $\Lambda^3(\mathbb{R}^7) = \mathbb{R}^1 \oplus \mathrm{S}_0(\mathbb{R}^7) \oplus \mathbb{R}^7$.

$G_2$ is the isotropy group of a generic element of $\omega \in \Lambda^3(\mathbb{R}^7)$:

$$G_2 \;=\; \{A \in \mathrm{SO}(7) \mid A \cdot \omega = \omega\}.$$

**Thm.** A 7-dimensional Riemannian mfd $(M^7, g, \omega)$ with a fixed $G_2$ structure $\omega \in \Lambda^3(M^7)$ has a $G_2$-invariant characteristic connection $\nabla^c$

$\Leftrightarrow$ the $\mathfrak{g}_2$ component of $\Gamma$ vanishes

$\Leftrightarrow$ There exists a VF $\beta$ with $\delta\omega = -\beta \lrcorner \omega$

The torsion of $\nabla^c$ is then $T^c = - * d\omega - \frac{1}{6}(d\omega, *\omega)\omega + *(\beta \wedge \omega)$ , and $\nabla^c$ admits (at least) one parallel spinor.

**Examples:** Explicit constructions of $G_2$ structures:

[Friedrich-Kath, Fernandez-Gray, Fernandez-Ugarte, Aloff-Wallach, Boyer-Galicki. . . ]

$M^7$: 3-Sasaki mnfd, corresponds to $\mathrm{SU}(2) \subset G_2 \subset \mathrm{SO}(7)$.

- Has 3 compatible contact structures $\eta_i \in T^*M^7$ and 3 Killing spinors $\psi_i \Rightarrow$ Ansatz:

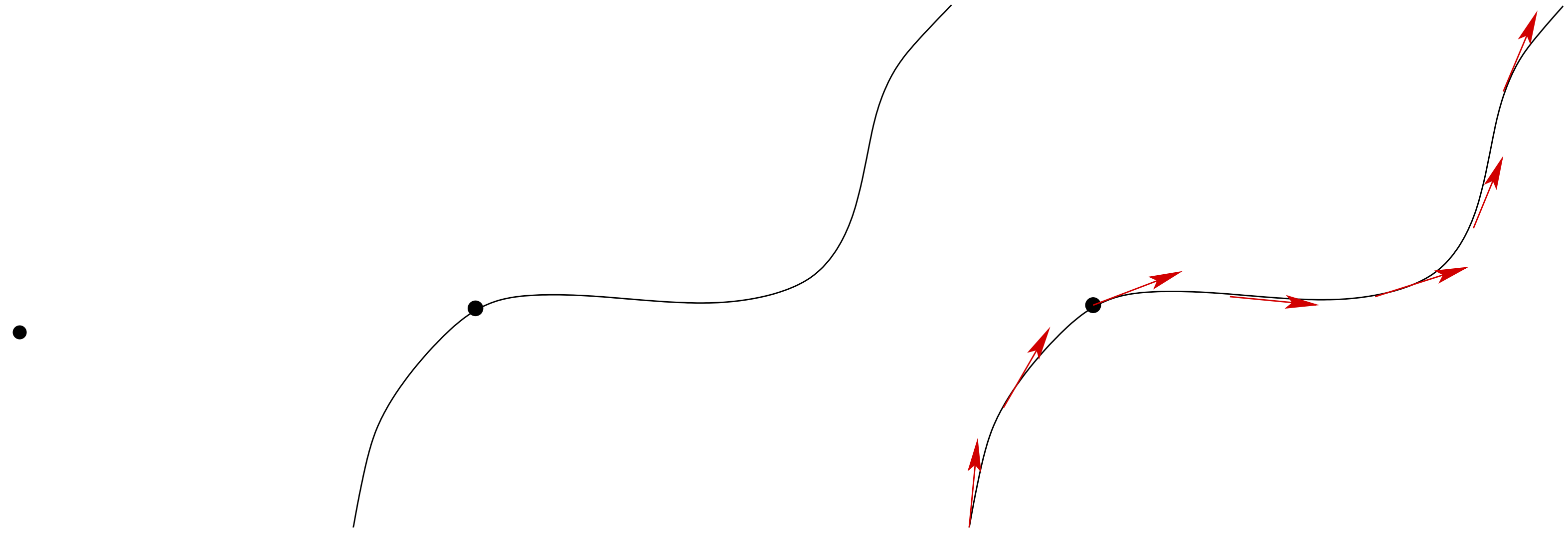
$$T = \sum_{i,j=1}^{3} \alpha_{ij}\eta_i \wedge d\eta_j + \gamma\eta_1 \wedge \eta_2 \wedge \eta_3, \quad \psi = \sum_{i=1}^{3} \mu_i\psi_i.$$

**Thm.** Every 7-dimensional 3-Sasaki mnfd admits a $\mathbb{P}^2$-family of metric connections with antisymmetric torsion and parallel spinors. Its holonomy is $G_2$. [IA-Friedrich, 2005]

$\Rightarrow$ First constructive global existence thm for parallel spinors!

We know show the relevance of parallel spinors for physics:

# Classical general relativity and electromagnetism



point particle

moves along a curve $\gamma$

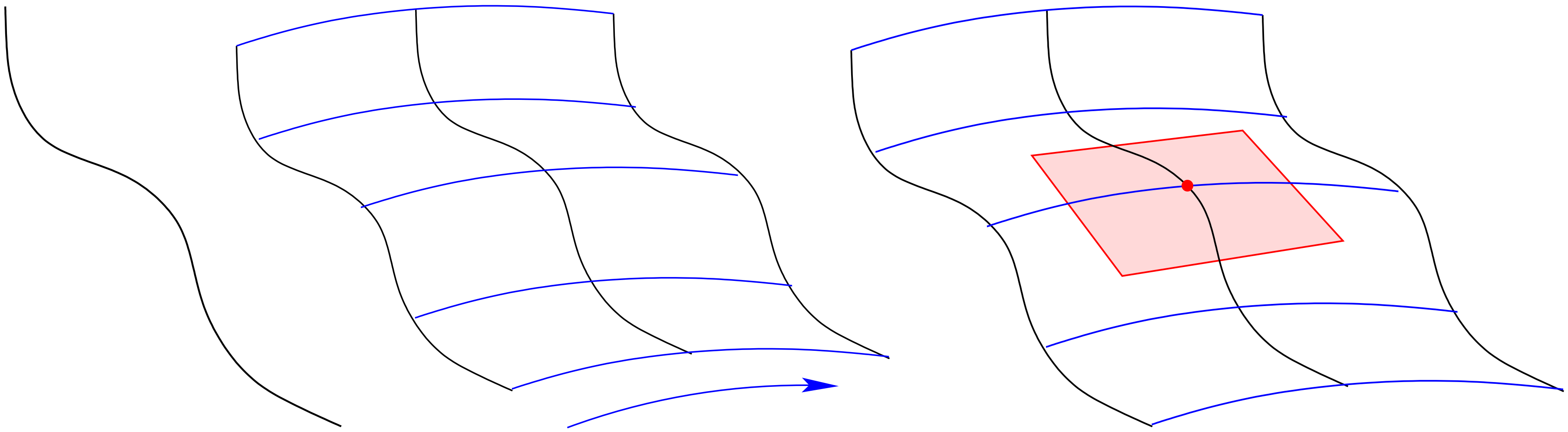
physical action: $\int_\gamma A$

for a potential $A$ : 1-form

field strength $F = dA$ : 2-form $\Leftrightarrow$ geometric concept of curvature

curvature measures deviation from vacuum !

# Modern unified models



string particle

moves along
a surface $S$

physical action: $\int_S \tilde{A}$ for
a higher order potential $\tilde{A}$ : 2-Form

higher order field strength
$F = d\tilde{A}$ : 3-form

$\Leftrightarrow$

geometric concept
of torsion

torsion measures deviation from vacuum ("integrable case") !

## Mathematical scheme for unified theories

No more described as Yang-Mills theories (electrodynamics, standard model of elementary particles), but rather:

• Particles are "oscillatory states" on some high dimensional configuration space

$$Y^{10,11} = V^{3\text{-}5} \times M^{5\text{-}8}$$

$V$: configuration space visible to the outside, i. e. Minkowski space or some solution from General Relativity (adS is popular here).

$M$: configuration space of *internal symmetries* $=$ Riemannian manifold with special geometric structure, quantized internal symmetries are described by spinor fields.

Example: Supersymmetry transformation, transform bosons into fermions and vice versa by tensoring with a (special) spin $1/2$ field ('Killing spinor').

[ $>$ 1980 Nieuwenhuizen, Strominger, Witten, Seiberg. . . ]

## Common sector of Type II string equations

- A. Strominger, 1986: $(M^n, g)$ Riemannian Spin mnfd with

  a $3$-Form $T$, a spinor field $\Psi$, and a function $\Phi$.
  (field strength) (supersymmetry) (dilaton)

  If one considers the metric connection $\nabla$ with torsion $T$, the field eqs. become:

- Bosonic eq.: $\mathrm{Ric}^{\nabla} + \frac{1}{2}\delta(T) + 2\mathrm{Hess}\,\Phi \;=\; 0, \quad \delta(e^{-2\Phi}T) \;=\; 0.$
- Fermionic eq.: $\nabla\Psi \;=\; 0, \quad T\cdot\Psi \;=\; 2\,d\Phi\cdot\Psi.$

  **Remarks:**

  - Bosonic eq. generalizes Einstein's eq. of general relativity
  - Calabi-Yau and parallel $G_2$ or $\mathrm{Spin}(7)$ mfds ($n = 7, 8$) are exact solution with $T = 0$ and $\Phi =$ const $\longrightarrow$ Bergers' list + algebraic geometry
  - For $T \neq 0$, the relation between curvature and spinor is subtler
  - $\exists$ models with higher order forms

# Main non existence theorem

**Thm.** A full solution of Strominger's model with $\Phi = \text{const}$ satisfies necessarily $T = 0$ or $\Psi = 0$.

[IA – $M$ compact, 2002, general case: IA-Friedrich-Nagy-Puhle, 2005]

**N.B.** Need only $\mathrm{Scal}^\nabla = 0$, not $\mathrm{Ric}^\nabla = 0$

$\Rightarrow$ physical corrections or deeper meaning of the dilaton

- $\exists$ solutions for any $3$ out of the $4$ equations
- Particularly interesting: solutions of $\nabla\Psi = 0$ (supersymmetries)

**Thm.** On a naturally reductive space $M = G/H$ with $\Phi = \text{const}$, any solution with $\nabla\Psi = 0$ and $T \cdot \Psi = 0$ satisfies $T = 0$ or $\Psi = 0$. [IA, 2002]

**N.B.** Proofs make heavy use of Dirac operators with torsion and their Weitzenböck formulas

**Thm.** Let $M$ be a *compact*, Ricci-flat manifold from Berger's list, $\psi \neq 0$ a $\nabla$-parallel spinor for some $T \in \Lambda^3(M)$ s. t. $\langle dT \cdot \psi, \psi \rangle \leq 0$. Then $T = 0$, i. e. *only* $\nabla^g$ can have parallel spinors. [IA-Friedrich, 2004]

– Physics interpretation: compact vacuum solutions are 'rigid' –

Different situation if $M^n$ is not compact:

Consider solvmanifolds $Y^7 = N \times \mathbb{R}$, $\mathfrak{n}$ : nilpotent 6-dim. Lie algebra $(\neq \mathfrak{h}_3 \oplus \mathfrak{h}_3) \Rightarrow$

1) $N$ carries "half flat" $\mathrm{SU}(3)$ structure,

2) $Y$ carries a $G_2$ structure $(\omega, g)$ with characteristic torsion $\neq 0$,

3) $Y$ carries – after a conformal change of the metric – an *integrable* $G_2$ structure $(\tilde{\omega}, \tilde{g})$. In particular, $\tilde{g}$ is Ricci flat und admits (at least) one LC-parallel spinor.

[Gibbons, Lü, Pope, Stelle (2002): described such a metric in local coordinates]

[Heber (1998): noncompact Einstein manifolds]

[Chiossi, Fino (2004): classification of all such solvmnfds (6 cases)]

[Hitchin (2001): existence of conformal change 3)]

**Thm.** For $\mathfrak{n} \cong (0, 0, e_{15}, e_{25}, 0, e_{12})$, there exists on $(Y, \tilde{\omega}, \tilde{g})$ a 1-parametric family $(T_h, \psi_h) \in \Lambda^3(Y) \times S(Y)$ s. t. every connection $\nabla^h$ with torsion $T_h$ satisfies:

$$\nabla^h \psi_h \ = \ 0.$$

For $h = 1$: $T_h = 0, \nabla^h = \nabla^g$ und $\psi_h$ coincides with the LC-parallel spinor. [IA-Chiossi-Fino, 2006]

– Only example of a Riemannian mnfd carrying a Ricci-flat integrable and a non-integrable geometry! –