The Classifying Lie Algebroid of a Geometric Structure

Ivan Struchiner

Universiteit Utrecht
Joint work with Prof. Rui Loja Fernandes

A Very Simple Problem

Metrics of Constant Curvature on $\mathbb{R}^2$

1. Describe the Moduli Space of germs of metrics in $\mathbb{R}^2$ with constant Gaussian curvature.
2. Describe their Symmetry Lie Algebras.
3. Construct examples of each of the metrics.

Solution to (1) and (2)

There is a bundle of Lie algebras $A \to X$, where:

- $X \equiv \mathbb{R}$ is the Moduli Space of germs of metrics of constant curvature on $\mathbb{R}^2$.

Solution to (1) and (2)

There is a bundle of Lie algebras $A \to X$, where:

- $X \equiv \mathbb{R}$ is the Moduli Space of germs of metrics of constant curvature on $\mathbb{R}^2$.
- $A \equiv X \times \mathbb{R}^3$ is a trivial vector bundle whose fibers can be identified with:

$\mathfrak{sl}_2$	if $k < 0$	Hyperbolic Geometry
$\mathfrak{se}_2$	if $k = 0$	Euclidean Geometry
$\mathfrak{so}_3$	if $k > 0$	Spherical Geometry

Solution to (3)

Let $\mathcal{G} \rightrightarrows X$ be the bundle of Lie groups with fibers:

$$\mathbf{s}^{-1}(k) = \begin{cases} \mathrm{SL}_2 \text{ if } k < 0 \\ \mathrm{SE}_2 \text{ if } k = 0 \\ \mathrm{SO}_3 \text{ if } k > 0 \end{cases}$$

Solution to (3)

Let $\mathcal{G} \rightrightarrows X$ be the bundle of Lie groups with fibers:

$$\mathbf{s}^{-1}(k) = \begin{cases} \mathrm{SL}_2 \text{ if } k < 0 \\ \mathrm{SE}_2 \text{ if } k = 0 \\ \mathrm{SO}_3 \text{ if } k > 0 \end{cases}$$

Each fiber of $\mathcal{G}$ can be identified with the (oriented) orthogonal frame bundle of a Riemannian 2-manifold:

$$M^2 = \begin{cases} \mathrm{SL}_2/\mathrm{SO}_2 \text{ if } k < 0 \\ \mathrm{SE}_2/\mathrm{SO}_2 \text{ if } k = 0 \\ \mathrm{SO}_3/\mathrm{SO}_2 \text{ if } k > 0 \end{cases}$$

Why is there a Lie Algebroid Associated to this Problem?

- Every metric can be characterized by a coframe on some other manifold.

Why is there a Lie Algebroid Associated to this Problem?

- Every metric can be characterized by a coframe on some other manifold.
- The moduli space of germs of metrics of constant Gaussian curvature in $\mathbb{R}^2$ depends on a finite amount of invariants (in this case only 1: the Gaussian curvature).

Equivalence of Coframes

- $\{\theta^1, \ldots, \theta^n\}$ a coframe on M.
- $\{\bar{\theta}^1, \ldots, \bar{\theta}^n\}$ a coframe on $\bar{M}$.

Definition

$\{\theta^i\}$ is (locally) equivalent to $\{\bar{\theta}^i\}$ if there exists a (locally defined) diffeomorphism

$$\phi : M \to \bar{M}$$

such that

$$\phi^*\bar{\theta}^i = \theta^i \text{ for all } i = 1, \ldots, n.$$

Invariants of Coframes

- $\mathrm{d}\theta^k = \sum C_{ij}^k(x)\theta^i \wedge \theta^j$
- $\mathrm{d}\bar{\theta}^k = \sum \bar{C}_{ij}^k(\bar{x})\bar{\theta}^i \wedge \bar{\theta}^j$

Invariants of Coframes

- $\mathrm{d}\theta^k = \sum C^k_{ij}(x)\theta^i \wedge \theta^j$
- $\mathrm{d}\bar{\theta}^k = \sum \bar{C}^k_{ij}(\bar{x})\bar{\theta}^i \wedge \bar{\theta}^j$

If $\{\theta^i\}$ is equivalent to $\{\bar{\theta}^i\}$ through $\phi : M \to \bar{M}$, then

$$\bar{C}^k_{ij}(\phi(x)) = C^k_{ij}(x).$$

Invariants of Coframes

- $\mathrm{d}\theta^k = \sum C^k_{ij}(x)\theta^i \wedge \theta^j$
- $\mathrm{d}\bar{\theta}^k = \sum \bar{C}^k_{ij}(\bar{x})\bar{\theta}^i \wedge \bar{\theta}^j$

If $\{\theta^i\}$ is equivalent to $\{\bar{\theta}^i\}$ through $\phi : M \to \bar{M}$, then

$$\bar{C}^k_{ij}(\phi(x)) = C^k_{ij}(x).$$

Definition

A function $I \in \mathrm{C}^\infty(M)$ is an invariant of $\{\theta^i\}$ if

$$I \circ \phi = I \text{ for every symmetry } \phi \text{ of } \{\theta^i\}$$

Cartan's Realization Problem

Initial Data:

- An integer $n \in \mathbb{N}$,
- an open subset $X \subset \mathbb{R}^d$,
- functions $C_{ij}^k \in \mathrm{C}^\infty(X)$ $(1 \leq i, j, k \leq n)$,
- functions $F_i^a \in \mathrm{C}^\infty(X)$ $(1 \leq a \leq d)$;

Cartan's Realization Problem (continued)

Does there exist a realization?

- An n-dimensional manifold M,
- a coframe $\{\theta^i\}$ on M,
- a smooth map $h : M \to X$

Cartan's Realization Problem (continued)

Does there exist a realization?

- An n-dimensional manifold M,
- a coframe $\{\theta^i\}$ on M,
- a smooth map $h : M \to X$

such that

$$\mathrm{d}\theta^k = \sum_{i<j} C_{ij}^k(h)\theta^i \wedge \theta^j$$

$$\mathrm{d}h^a = \sum_i F_i^a(h)\theta^i.$$

Cartan's Realization Problem (continued)

Does there exist a realization?

- An n-dimensional manifold M,
- a coframe $\{\theta^i\}$ on M,
- a smooth map $h : M \to X$

such that

$$\mathrm{d}\theta^k = \sum_{i<j} C^k_{ij}(h)\theta^i \wedge \theta^j$$

$$\mathrm{d}h^a = \sum_i F^a_i(h)\theta^i.$$

Definition

A solution (M, θ^i, h) to Cartan's problem will be called a realization.

Morphisms of Realizations and the Local Classification Problem

Definition

Let (M_1, θ_1, h_1) and (M_2, θ_2, h_2) be two realizations of (n, X, C_{ij}^k, F_i^a). A morphism of realizations is a local diffeomorphism $\phi : M_1 \to M_2$ such that

$$\phi^*\theta_2 = \theta_1 \text{ and } h_2 \circ \phi = h_1.$$

Morphisms of Realizations and the Local Classification Problem

Definition

Let (M_1, θ_1, h_1) and (M_2, θ_2, h_2) be two realizations of (n, X, C_{ij}^k, F_i^a). A morphism of realizations is a local diffeomorphism $\phi : M_1 \to M_2$ such that

$$\phi^*\theta_2 = \theta_1 \text{ and } h_2 \circ \phi = h_1.$$

Local Classification Problem

- What are all the germs of solutions of a Cartan's realization problem?

Morphisms of Realizations and the Local Classification Problem

Definition

Let (M_1, θ_1, h_1) and (M_2, θ_2, h_2) be two realizations of (n, X, C_{ij}^k, F_i^a). A morphism of realizations is a local diffeomorphism $\phi : M_1 \to M_2$ such that

$$\phi^*\theta_2 = \theta_1 \text{ and } h_2 \circ \phi = h_1.$$

Local Classification Problem

- What are all the germs of solutions of a Cartan's realization problem?
- When are two germs of realizations isomorphic?

Another Very Simple Example: $d = o$

- Suppose that $X = \{pt\}$, i.e., $d = 0$.

Another Very Simple Example: $d = o$

- Suppose that $X = \{pt\}$, i.e., $d = 0$.
- Thus, C_{ij}^{k} are all constant and $F_i^a \equiv 0$.

Another Very Simple Example: $d = o$

- Suppose that $X = \{pt\}$, i.e., $d = 0$.
- Thus, C_{ij}^k are all constant and $F_i^a \equiv 0$.
- Necessary conditions for solving the realization problem are obtained from $\mathrm{d}^2 = 0$: $-C_{ij}^k$ must be the structure constants of a Lie algebra $\mathfrak{g}$.

Another Very Simple Example: $d = o$

- Suppose that $X = \{pt\}$, i.e., $d = 0$.
- Thus, C_{ij}^k are all constant and $F_i^a \equiv 0$.
- Necessary conditions for solving the realization problem are obtained from $\mathrm{d}^2 = 0$: $-C_{ij}^k$ must be the structure constants of a Lie algebra $\mathfrak{g}$.
- This condition is also sufficient: $(G, \omega_{\mathrm{MC}})$ is a realization.

Another Very Simple Example: $d = o$

- Suppose that $X = \{pt\}$, i.e., $d = 0$.
- Thus, C_{ij}^k are all constant and $F_i^a \equiv 0$.
- Necessary conditions for solving the realization problem are obtained from $\mathrm{d}^2 = 0$: $-C_{ij}^k$ must be the structure constants of a Lie algebra $\mathfrak{g}$.
- This condition is also sufficient: $(G, \omega_{\mathrm{MC}})$ is a realization.
- The universal property of the Maurer-Cartan form implies that locally this is the only solution.

Another Very Simple Example: $d = o$

- Suppose that $X = \{pt\}$, i.e., $d = 0$.
- Thus, C^k_{ij} are all constant and $F^a_i \equiv 0$.
- Necessary conditions for solving the realization problem are obtained from $\mathrm{d}^2 = 0$: $-C^k_{ij}$ must be the structure constants of a Lie algebra $\mathfrak{g}$.
- This condition is also sufficient: $(G, \omega_{\mathrm{MC}})$ is a realization.
- The universal property of the Maurer-Cartan form implies that locally this is the only solution.

In general $(d \neq 0)$, we will have to consider Lie algebroids instead of Lie algebras.

Another Very Simple Example: $d = o$

- Suppose that $X = \{pt\}$, i.e., $d = 0$.
- Thus, C_{ij}^k are all constant and $F_i^a \equiv 0$.
- Necessary conditions for solving the realization problem are obtained from $\mathrm{d}^2 = 0$: $-C_{ij}^k$ must be the structure constants of a Lie algebra $\mathfrak{g}$.
- This condition is also sufficient: $(G, \omega_{\mathrm{MC}})$ is a realization.
- The universal property of the Maurer-Cartan form implies that locally this is the only solution.

In general $(d \neq 0)$, we will have to consider Lie algebroids instead of Lie algebras.

Lie Algebroids

Definition

A Lie algebroid is a vector bundle $A \to X$ equipped with a Lie bracket $[\ ,\ ]$ on $\Gamma(A)$ and a bundle map $\# : A \to TX$ such that:

$[\alpha, f\beta] = f[\alpha, \beta] + \#(\alpha)(f)\beta$ for all $\alpha, \beta \in \Gamma(A)$ and $f \in \mathrm{C}^\infty(X)$.

Existence of Solutions: Necessary Conditions

Proposition

Let (n, X, C^k_{ij}, F^a_i) be the initial data of a Cartan's realization problem. If for every $x_0 \in X$ there is a realization (M, θ, h) with $h(p_0) = x_0$, for some $p_0 \in M$, then the $-C^k_{ij}, F^a_i \in \mathrm{C}^\infty(X)$ are the structure functions of a Lie algebroid A over X.

Existence of Solutions: Necessary Conditions

Proposition

Let (n, X, C_{ij}^k, F_i^a) be the initial data of a Cartan's realization problem. If for every $x_0 \in X$ there is a realization (M, θ, h) with $h(p_0) = x_0$, for some $p_0 \in M$, then the $-C_{ij}^k, F_i^a \in \mathrm{C}^\infty(X)$ are the structure functions of a Lie algebroid A over X.

Proof

- $A = X \times \mathbb{R}^n \to X$, with basis of sections $\{\alpha_1, \ldots \alpha_n\}$ and coordinates $(x_1, \ldots, x_d)$ on X

Existence of Solutions: Necessary Conditions

Proposition

Let (n, X, C_{ij}^k, F_i^a) be the initial data of a Cartan's realization problem. If for every $x_0 \in X$ there is a realization (M, θ, h) with $h(p_0) = x_0$, for some $p_0 \in M$, then the $-C_{ij}^k, F_i^a \in \mathrm{C}^\infty(X)$ are the structure functions of a Lie algebroid A over X.

Proof

- $A = X \times \mathbb{R}^n \to X$, with basis of sections $\{\alpha_1, \ldots \alpha_n\}$ and coordinates $(x_1, \ldots, x_d)$ on X
- $[\alpha_i, \alpha_j](x) = \sum C_{ij}^k(x)\alpha_k$,

Existence of Solutions: Necessary Conditions

Proposition

Let (n, X, C_{ij}^k, F_i^a) be the initial data of a Cartan's realization problem. If for every $x_0 \in X$ there is a realization (M, θ, h) with $h(p_0) = x_0$, for some $p_0 \in M$, then the $-C_{ij}^k, F_i^a \in \mathrm{C}^\infty(X)$ are the structure functions of a Lie algebroid A over X.

Proof

- $A = X \times \mathbb{R}^n \to X$, with basis of sections $\{\alpha_1, \ldots \alpha_n\}$ and coordinates $(x_1, \ldots, x_d)$ on X
- $[\alpha_i, \alpha_j](x) = \sum C_{ij}^k(x)\alpha_k$, $\#(\alpha_i)(x) = \sum F_i^a(x)\frac{\partial}{\partial x_a}$

Existence of Solutions: Necessary Conditions

Proposition

Let (n, X, C_{ij}^k, F_i^a) be the initial data of a Cartan's realization problem. If for every $x_0 \in X$ there is a realization (M, θ, h) with $h(p_0) = x_0$, for some $p_0 \in M$, then the $-C_{ij}^k, F_i^a \in \mathrm{C}^\infty(X)$ are the structure functions of a Lie algebroid A over X.

Proof

- $A = X \times \mathbb{R}^n \to X$, with basis of sections $\{\alpha_1, \ldots \alpha_n\}$ and coordinates $(x_1, \ldots, x_d)$ on X
- $[\alpha_i, \alpha_j](x) = \sum C_{ij}^k(x)\alpha_k$, $\#(\alpha_i)(x) = \sum F_i^a(x)\frac{\partial}{\partial x_a}$
- $\mathrm{d}^2\theta^k = 0$ implies that $[\ ,\ ]$ satisfies the Jacobi Identity.

Existence of Solutions: Necessary Conditions

Proposition

Let (n, X, C_{ij}^k, F_i^a) be the initial data of a Cartan's realization problem. If for every $x_0 \in X$ there is a realization (M, θ, h) with $h(p_0) = x_0$, for some $p_0 \in M$, then the $-C_{ij}^k, F_i^a \in \mathrm{C}^\infty(X)$ are the structure functions of a Lie algebroid A over X.

Proof

- $A = X \times \mathbb{R}^n \to X$, with basis of sections $\{\alpha_1, \ldots \alpha_n\}$ and coordinates $(x_1, \ldots, x_d)$ on X
- $[\alpha_i, \alpha_j](x) = \sum C_{ij}^k(x)\alpha_k$, $\#(\alpha_i)(x) = \sum F_i^a(x)\frac{\partial}{\partial x_a}$
- $\mathrm{d}^2\theta^k = 0$ implies that $[\ ,\ ]$ satisfies the Jacobi Identity.
- $\mathrm{d}^2 h_a = 0$ implies that $\#$ is a Lie algebra homorphism.

Classifying Lie Algebroid

Definition

The Lie algebroid $A \to X$ is called the Classifying Lie Algebroid of the Cartan's realization problem.

Realizations and Lie Algebroid Morphisms

Each realization (M, θ, h) of the problem determines a bundle map:

$$\begin{tikzcd} TM \arrow[r, "\theta"] \arrow[d] & A \arrow[d] \\ M \arrow[r, "h"'] & X \end{tikzcd}$$

$$\begin{aligned} TM &\longrightarrow A = X \times \mathbb{R}^n, \\ v &\longmapsto (h(p(v)), (\theta^1(v), \dots, \theta^n(v))), \end{aligned}$$

Realizations and Lie Algebroid Morphisms

Proposition

Let $A \to X$ be the classifying Lie algebroid of a Cartan's problem. The realizations of this problem are in 1:1 correspondence with bundle maps as above which are Lie algebroid morphisms and fiberwise isomorphisms.

Realizations and Lie Algebroid Morphisms

Proposition

Let $A \to X$ be the classifying Lie algebroid of a Cartan's problem. The realizations of this problem are in 1:1 correspondence with bundle maps as above which are Lie algebroid morphisms and fiberwise isomorphisms.

Proof

- $\mathrm{d}h^a = \sum_i F_i^a(h)\theta^i$ if and only if (θ, h) is compatible with anchors.

Realizations and Lie Algebroid Morphisms

Proposition

Let $A \to X$ be the classifying Lie algebroid of a Cartan's problem. The realizations of this problem are in 1:1 correspondence with bundle maps as above which are Lie algebroid morphisms and fiberwise isomorphisms.

Proof

- $\mathrm{d}h^a = \sum_i F_i^a(h)\theta^i$ if and only if (θ, h) is compatible with anchors.
- $\mathrm{d}\theta^k = \sum_{i<j} C_{ij}^k(h)\theta^i \wedge \theta^j$ if and only if (θ, h) is compatible with brackets.

Maurer-Cartan Equation

Definition

A bundle map

$$\begin{tikzcd} TM \arrow[r, "\theta"] \arrow[d] & A \arrow[d] \\ M \arrow[r, "h"'] & X \end{tikzcd}$$

which is compatible with the anchors will be called a 1-form with values in A. We will write $\theta \in \Omega^1(M, A)$.

Maurer-Cartan Equation

Definition

A bundle map

$$\begin{tikzcd} TM \arrow[r, "\theta"] \arrow[d] & A \arrow[d] \\ M \arrow[r, "h"'] & X \end{tikzcd}$$

which is compatible with the anchors will be called a 1-form with values in A. We will write $\theta \in \Omega^1(M, A)$.

The bracket compatibility can be expressed in terms of a Maurer-Cartan equation:

$$\mathrm{d}_\nabla \theta + \frac{1}{2}[\theta, \theta]_\nabla = 0$$

The Maurer-Cartan Form on $\mathcal{G}$

Definition

The Maurer-Cartan form on a Lie groupoid $\mathcal{G}$ is the $\mathbf{s}$-foliated 1-forma with values in A

$$\begin{array}{ccc} T^{\mathbf{s}}\mathcal{G} & \xrightarrow{\omega_{\mathrm{MC}}} & A \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow[\mathbf{t}]{} & X \end{array}$$

defined by

$$\omega_{\mathrm{MC}}(\xi) = (\mathrm{d}R_{g^{-1}})_g(\xi) \in A_{\mathbf{t}(g)}$$

for all $\xi \in T^{\mathbf{s}}_g\mathcal{G}$.

The Maurer-Cartan Form on $\mathcal{G}$

Definition

The Maurer-Cartan form on a Lie groupoid $\mathcal{G}$ is the s-foliated 1-forma with values in A

$$\begin{tikzcd} T^{\mathbf{s}}\mathcal{G} \arrow[r, "\omega_{\mathrm{MC}}"] \arrow[d] & A \arrow[d] \\ \mathcal{G} \arrow[r, "\mathbf{t}"'] & X \end{tikzcd}$$

defined by

$$\omega_{\mathrm{MC}}(\xi) = (\mathrm{d}R_{g^{-1}})_g(\xi) \in A_{\mathbf{t}(g)}$$

for all $\xi \in T^{\mathbf{s}}_g\mathcal{G}$.

$$\mathrm{d}_\nabla \omega_{\mathrm{MC}} + \frac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}]_\nabla = 0$$

The Local Universal Property

Proposition

- $(\theta, h) \in \Omega^1(M, A)$

The Local Universal Property

Proposition

- $(\theta, h) \in \Omega^1(M, A)$
- $A \to X$ a Lie algebroid integrated by $\mathcal{G} \rightrightarrows X$

The Local Universal Property

Proposition

- $(\theta, h) \in \Omega^1(M, A)$
- $A \to X$ a Lie algebroid integrated by $\mathcal{G} \rightrightarrows X$
- $\mathrm{rank}(A) = \dim M$

The Local Universal Property

Proposition

- $(\theta, h) \in \Omega^1(M, A)$
- $A \to X$ a Lie algebroid integrated by $\mathcal{G} \rightrightarrows X$
- $\mathrm{rank}(A) = \dim M$
- $\mathrm{d}_\nabla \theta + \frac{1}{2}[\theta, \theta]_\nabla = 0.$

The Local Universal Property

Proposition

- $(\theta, h) \in \Omega^1(M, A)$
- $A \to X$ a Lie algebroid integrated by $\mathcal{G} \rightrightarrows X$
- $\mathrm{rank}(A) = \dim M$
- $\mathrm{d}_\nabla \theta + \frac{1}{2}[\theta, \theta]_\nabla = 0.$

Then, for every $p \in M$ and $g \in \mathcal{G}$ such that $h(p) = \mathbf{t}(g)$, there exists a unique locally defined diffeomorphism $\phi : M \to \mathbf{s}^{-1}(\mathbf{s}(g))$ such that

$$\begin{cases} \phi(p) & = & g \\ \phi^*\omega_{\mathrm{MC}} & = & \theta. \end{cases}$$

The Local Universal Property

Proposition

- $(\theta, h) \in \Omega^1(M, A)$
- $A \to X$ a Lie algebroid integrated by $\mathcal{G} \rightrightarrows X$
- $\mathrm{rank}(A) = \dim M$
- $\mathrm{d}_\nabla \theta + \frac{1}{2}[\theta, \theta]_\nabla = 0.$

Then, for every $p \in M$ and $g \in \mathcal{G}$ such that $h(p) = \mathbf{t}(g)$, there exists a unique locally defined diffeomorphism $\phi : M \to \mathbf{s}^{-1}(\mathbf{s}(g))$ such that

$$\begin{cases} \phi(p) & = & g \\ \phi^* \omega_{\mathrm{MC}} & = & \theta. \end{cases}$$

Corollary of the Uniqueness

If $\phi : \mathbf{s}^{-1}(x) \to \mathbf{s}^{-1}(y)$ is a symmetry of ω_{MC} then ϕ is locally of the form $\phi = R_g$ for some $g \in \mathcal{G}$.

Solution to the Local Classification Problem

- Necessary condition for existence of realizations: $-C_{ij}^k, F_i^a$ determine a classifying Lie algebroid $A \to X$.

Solution to the Local Classification Problem

- Necessary condition for existence of realizations: $-C_{ij}^k, F_i^a$ determine a classifying Lie algebroid $A \to X$.
- Let $\mathcal{G}$ be a (local) Lie groupoid integrating A.

Solution to the Local Classification Problem

- Necessary condition for existence of realizations: $-C_{ij}^k, F_i^a$ determine a classifying Lie algebroid $A \to X$.
- Let $\mathcal{G}$ be a (local) Lie groupoid integrating A.
- For each $x_0 \in X$, $(\mathbf{s}^{-1}(x_0), \omega_{\mathrm{MC}}, \mathbf{t})$ is a realization with $\mathbf{t}(\mathbf{1}_{x_0}) = x_0$.

Solution to the Local Classification Problem

- Necessary condition for existence of realizations: $-C_{ij}^k, F_i^a$ determine a classifying Lie algebroid $A \to X$.
- Let $\mathcal{G}$ be a (local) Lie groupoid integrating A.
- For each $x_0 \in X$, $(\mathbf{s}^{-1}(x_0), \omega_{\mathrm{MC}}, \mathbf{t})$ is a realization with $\mathbf{t}(\mathbf{1}_{x_0}) = x_0$.
- Locally, these are the only realizations (universal property).

Symmetries of a Realization

Definition

- A symmetry of (M, θ, h) is a diffeomorphism $\phi : M \to M$ such that
$$\phi^*\theta = \theta \text{ and } h \circ \phi = h.$$

Symmetries of a Realization

Definition

- A symmetry of (M, θ, h) is a diffeomorphism $\phi : M \to M$ such that
$$\phi^*\theta = \theta \text{ and } h \circ \phi = h.$$
- An infinitesimal symmetry is a vector field $\xi \in \mathfrak{X}(M)$ such that
$$\mathcal{L}_\xi \theta = 0 \text{ and } \mathcal{L}_\xi h = 0.$$

Symmetries of a Realization

Definition

- A symmetry of (M, θ, h) is a diffeomorphism $\phi : M \to M$ such that
$$\phi^*\theta = \theta \text{ and } h \circ \phi = h.$$

- An infinitesimal symmetry is a vector field $\xi \in \mathfrak{X}(M)$ such that
$$\mathcal{L}_\xi\theta = 0 \text{ and } \mathcal{L}_\xi h = 0.$$

- $\mathfrak{X}(M, \theta, h)_p$ denotes the germs of infinitesimal symmetries of (M, θ, h) at a point $p \in M$.

Symmetries of a Realization (continued)

Proposition

- (M, θ, h): Realization of a Cartan's problem.
- $A \to X$: Classifying Lie algebroid.
- $\mathfrak{g}_x$: Isotropy Lie algebra of A at $x \in X$.

Symmetries of a Realization (continued)

Proposition

- (M, θ, h): Realization of a Cartan's problem.
- $A \to X$: Classifying Lie algebroid.
- $\mathfrak{g}_x$: Isotropy Lie algebra of A at $x \in X$.

Then $\mathfrak{X}(M, \theta, h)_p$ is a Lie algebra isomorphic to $\mathfrak{g}_{h(p)}$ and in particular

$$\dim \mathfrak{X}(M, \theta, h)_p = \dim M - \dim L_{h(p)}.$$

Coverings of Realizations and Global Equivalence

Definition

- A Covering of realizations is a surjective morphism of realizations.

Coverings of Realizations and Global Equivalence

Definition

- A Covering of realizations is a surjective morphism of realizations.
- (M_1, θ_1, h) and (M_2, θ_2, h_2) are globally equivalent up to covering, if there exists another realization (M, θ, h) which covers both M_1 and M_2:

$$\begin{tikzcd} & (M, \theta, h) \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ (M_1, \theta_1, h_1) & & (M_2, \theta_2, h_2). \end{tikzcd}$$

The Global Classification Problem and the Globalization Problem

Global Classification Problem

What are all the solutions of a Cartan's realization problem up to global equivalence, up to covering?

The Global Classification Problem and the Globalization Problem

Global Classification Problem

What are all the solutions of a Cartan's realization problem up to global equivalence, up to covering?

Globalization Problem

When are two germs of realizations contained in the same connected realization?

Solution to the Globalization Problem

Theorem

Suppose that the classifying Lie algebroid $A \to X$ is integrable. Then (θ_0, h_0) and (θ_1, h_1) are germs of the same connected realization (M, θ, h) if and only if they correspond to points of X in the same orbit of A.

Solution to the Globalization Problem

Theorem

Suppose that the classifying Lie algebroid $A \to X$ is integrable. Then (θ_0, h_0) and (θ_1, h_1) are germs of the same connected realization (M, θ, h) if and only if they correspond to points of X in the same orbit of A.

- If A is not integrable then the theorem may not be true.

Solution to the Global Classification Problem

Theorem

- (M, θ, h): realization of a Cartan's problem.
- $A \to X$: integrable.

Solution to the Global Classification Problem

Theorem

- (M, θ, h): realization of a Cartan's problem.
- $A \to X$: integrable.

Then M is globally equivalent up to cover to an open set of an s-fiber of a groupoid $\mathcal{G}$ integrating A.

Global Classification II

Corollary

Rank 0 If the structure functions of a coframe θ on M are constant, then M is globally equivalent, up to covering, to an open set of a Lie group.

Global Classification II

Corollary

Rank 0: If the structure functions of a coframe θ on M are constant, then M is globally equivalent, up to covering, to an open set of a Lie group.

Rank n:

- (M_1, θ_1, h_1) and (M_2, θ_2, h_2): realizations of a Cartan's problem.
- $h_1(M_1) = h_2(M_2) \subset L$.
- $L \subset X$: an n-dimensional orbit of A.

Global Classification II

Corollary

Rank 0 If the structure functions of a coframe θ on M are constant, then M is globally equivalent, up to covering, to an open set of a Lie group.

Rank n

- (M_1, θ_1, h_1) and (M_2, θ_2, h_2): realizations of a Cartan's problem.
- $h_1(M_1) = h_2(M_2) \subset L$.
- $L \subset X$: an n-dimensional orbit of A.

Then (L, θ, h) is a realization of the Cartan's problem covered by both (M_1, θ_1, h_1) and (M_2, θ_2, h_2), i.e.,

$$\begin{tikzcd} M_1 \arrow[dr, "h_1"'] & & M_2 \arrow[dl, "h_2"] \\ & L & \end{tikzcd}$$

Special Symplectic Geometries

Studied extensively by Schwachhöfer, Cahen, Chi, Merkulov, Bryant,.....

Special Symplectic Geometries

Studied extensively by Schwachhöfer, Cahen, Chi, Merkulov, Bryant,.....

I) Bochner-bi-Lagrangian M is a symplectic manifold equipped with a pair of complementary Lagrangian distributions which are parallel and the Bochner tensor of the curvature vanishes.

Special Symplectic Geometries

Studied extensively by Schwachhöfer, Cahen, Chi, Merkulov, Bryant,.....

I) Bochner-bi-Lagrangian M is a symplectic manifold equipped with a pair of complementary Lagrangian distributions which are parallel and the Bochner tensor of the curvature vanishes.

II) Bochner-Kähler M is a Kähler manifold for which the Bochner tensor of the curvature vanishes.

Special Symplectic Geometries

Studied extensively by Schwachhöfer, Cahen, Chi, Merkulov, Bryant,.....

I) Bochner-bi-Lagrangian M is a symplectic manifold equipped with a pair of complementary Lagrangian distributions which are parallel and the Bochner tensor of the curvature vanishes.

II) Bochner-Kähler M is a Kähler manifold for which the Bochner tensor of the curvature vanishes.

III) Ricci Type M is a symplectic manifold equipped with a symplectic connection for which the Ricci flat component of its curvature vanishes.

Special Symplectic Geometries

Studied extensively by Schwachhöfer, Cahen, Chi, Merkulov, Bryant,.....

I) Bochner-bi-Lagrangian M is a symplectic manifold equipped with a pair of complementary Lagrangian distributions which are parallel and the Bochner tensor of the curvature vanishes.

II) Bochner-Kähler M is a Kähler manifold for which the Bochner tensor of the curvature vanishes.

III) Ricci Type M is a symplectic manifold equipped with a symplectic connection for which the Ricci flat component of its curvature vanishes.

IV) Proper Symplectic Holonomy M is symplectic manifold equipped with a symplectic connection whose holonomy group is a proper irreducible subgroup of $\mathrm{Sp}(V)$.

Special Symplectic Lie Algebras

All of these geometries are linked by their structure groups.

Special Symplectic Lie Algebras

All of these geometries are linked by their structure groups.

- If $\mathfrak{h}$ is a subalgebra of $\mathfrak{sp}(V, \omega_0)$ then there is a bilinear map $\circ : S^2(V) \cong \mathfrak{sp}(V) \to \mathfrak{h}$

$$(u \circ v, T) = \omega_0(Tu, v) \text{ for all } u, v \in V \text{ and } T \in \mathfrak{h}.$$

Special Symplectic Lie Algebras

All of these geometries are linked by their structure groups.

- If $\mathfrak{h}$ is a subalgebra of $\mathfrak{sp}(V, \omega_0)$ then there is a bilinear map $\circ : S^2(V) \cong \mathfrak{sp}(V) \to \mathfrak{h}$

$$(u \circ v, T) = \omega_0(Tu, v) \text{ for all } u, v \in V \text{ and } T \in \mathfrak{h}.$$

Definition

$\mathfrak{h}$ is a special symplectic Lie algebras if

$$(u \circ v)w - (u \circ w)v = 2\omega_0(v, w)u - \omega_0(u, v)w + \omega_0(u, w)v.$$

Special Symplectic Lie Algebras

All of these geometries are linked by their structure groups.

- If $\mathfrak{h}$ is a subalgebra of $\mathfrak{sp}(V, \omega_0)$ then there is a bilinear map $\circ : S^2(V) \cong \mathfrak{sp}(V) \to \mathfrak{h}$

$$(u \circ v, T) = \omega_0(Tu, v) \text{ for all } u, v \in V \text{ and } T \in \mathfrak{h}.$$

Definition

$\mathfrak{h}$ is a special symplectic Lie algebras if

$$(u \circ v)w - (u \circ w)v = 2\omega_0(v, w)u - \omega_0(u, v)w + \omega_0(u, w)v.$$

- These are the Berger algebras of torsion-free symplectic connections

Special Symplectic Lie Algebras

All of these geometries are linked by their structure groups.

- If $\mathfrak{h}$ is a subalgebra of $\mathfrak{sp}(V,\omega_0)$ then there is a bilinear map $\circ : S^2(V) \cong \mathfrak{sp}(V) \to \mathfrak{h}$

$$(u \circ v, T) = \omega_0(Tu, v) \text{ for all } u, v \in V \text{ and } T \in \mathfrak{h}.$$

Definition

$\mathfrak{h}$ is a special symplectic Lie algebras if

$$(u \circ v)w - (u \circ w)v = 2\omega_0(v,w)u - \omega_0(u,v)w + \omega_0(u,w)v.$$

- These are the Berger algebras of torsion-free symplectic connections
- These Lie algebras have been classified.

Special Symplectic Manifolds

- There is an injective map $\mathfrak{h} \to \mathcal{K}(\mathfrak{h})$:

$$T \mapsto R_T(u,v) \mapsto 2\omega_0(u,v)T + u \circ (Tv) - v \circ (Tu).$$

$$R_T(u,v) = 2\omega_0(u,v)T + u \circ (Tv) - v \circ (Tu),$$

where

$$\mathcal{K}(\mathfrak{h}) = \left\{R \in \wedge^2 V^* \otimes \mathfrak{g} : R(u,v)w + \text{cycl. perm.} = 0, \text{ for all } u,v,w \in V\right\}$$

Special Symplectic Manifolds

- There is an injective map $\mathfrak{h} \to \mathcal{K}(\mathfrak{h})$:

$$T \mapsto R_T(u,v) \mapsto 2\omega_0(u,v)T + u \circ (Tv) - v \circ (Tu).$$

$$R_T(u,v) = 2\omega_0(u,v)T + u \circ (Tv) - v \circ (Tu),$$

where

$$\mathcal{K}(\mathfrak{h}) = \left\{R \in \wedge^2 V^* \otimes \mathfrak{g} : R(u,v)w + \text{cycl. perm.} = 0, \text{ for all } u,v,w \in V\right\}$$

- Set: $\mathcal{R}_{\mathfrak{h}} = \{R_T : T \in \mathfrak{h}\} \cong \mathfrak{h}$

Special Symplectic Manifolds

- There is an injective map $\mathfrak{h} \to \mathcal{K}(\mathfrak{h})$:

$$T \mapsto R_T(u,v) \mapsto 2\omega_0(u,v)T + u \circ (Tv) - v \circ (Tu).$$

$$R_T(u,v) = 2\omega_0(u,v)T + u \circ (Tv) - v \circ (Tu),$$

where

$$\mathcal{K}(\mathfrak{h}) = \left\{R \in \wedge^2 V^* \otimes \mathfrak{g} : R(u,v)w + \text{cycl. perm.} = 0, \text{ for all } u,v,w \in V\right\}$$

- Set: $\mathcal{R}_{\mathfrak{h}} = \{R_T : T \in \mathfrak{h}\} \cong \mathfrak{h}$

Definition

A torsion-free connection $\eta \in \Omega^1(\mathcal{B}_{\mathrm{Sp_n}}(M), \mathfrak{sp}_n)$ on (M,ω) is a special symplectic connection associated to $\mathfrak{h} \subset \mathfrak{sp}(V)$ if its curvature takes values in $\mathcal{R}_{\mathfrak{h}}$.

Structure Equations

Theorem (Cahen e Schwachhöfer)

Let (M, ω, η) be a special symplectic manifold associated to $\mathfrak{h}$. Then there exists:

- an H-structure over M which is compatible with η

Structure Equations

Theorem (Cahen e Schwachhöfer)

Let (M, ω, η) be a special symplectic manifold associated to $\mathfrak{h}$. Then there exists:

- an H-structure over M which is compatible with η
- maps $\rho : \mathcal{B}_H(M) \to \mathfrak{h}$, $u : \mathcal{B}_H(M) \to V$ and $f : \mathcal{B}_H(M) \to \mathbb{R}$

such that

$$\begin{cases} \mathrm{d}\theta &= -\eta \wedge \theta \\ \mathrm{d}\eta &= R_\rho(\theta \wedge \theta) - \eta \wedge \eta \\ \mathrm{d}\rho &= u \circ \theta - [\eta, \rho] \\ \mathrm{d}u &= (\rho^2 + f)\theta - \eta u \\ \mathrm{d}f &= -2\omega(\rho u, \theta) \quad (= -\mathrm{d}(\rho, \rho)) \end{cases}$$

where

$$R_\rho(x, y) = 2\omega(x, y)\rho + x \circ (\rho y) - y \circ (\rho x).$$

The Classifying Lie Algebroid

- $X \cong \mathfrak{h} \oplus V \oplus \mathbb{R}$

The Classifying Lie Algebroid

- $X \cong \mathfrak{h} \oplus V \oplus \mathbb{R}$
- $A = X \times (V \oplus \mathfrak{h})$

The Classifying Lie Algebroid

- $X \cong \mathfrak{h} \oplus V \oplus \mathbb{R}$
- $A = X \times (V \oplus \mathfrak{h})$
- Bracket:

$$[(x, T), (y, U)](\rho, u, f) = (Ty - Ux, [T, U] - R_\rho(x, y))$$

The Classifying Lie Algebroid

- $X \cong \mathfrak{h} \oplus V \oplus \mathbb{R}$
- $A = X \times (V \oplus \mathfrak{h})$
- Bracket:

$$[(x, T), (y, U)](\rho, u, f) = (Ty - Ux, [T, U] - R_\rho(x, y))$$

- Anchor:

$$\#(x, T)(\rho, u, f) = (u \circ x - [T, \rho], (\rho^2 + f)x - Tu, -2\omega(\rho u, x))$$

Associated Poisson Manifold

- $F_c = \{(\rho, u, f) \in \mathfrak{h} \oplus V \oplus \mathbb{R} : f + (\rho, \rho) = c\} \subset \mathfrak{h} \oplus V \oplus \mathbb{R}$

Associated Poisson Manifold

- $F_c = \{(\rho, u, f) \in \mathfrak{h} \oplus V \oplus \mathbb{R} : f + (\rho, \rho) = c\} \subset \mathfrak{h} \oplus V \oplus \mathbb{R}$
- The restriction $A_c = A|_{F_c}$ is the cotangent Lie algebroid of a Poisson manifold.

Associated Poisson Manifold

- $F_c = \{(\rho, u, f) \in \mathfrak{h} \oplus V \oplus \mathbb{R} : f + (\rho, \rho) = c\} \subset \mathfrak{h} \oplus V \oplus \mathbb{R}$
- The restriction $A_c = A|_{F_c}$ is the cotangent Lie algebroid of a Poisson manifold.
- All of these Poisson structures may be put together into an integrable Poisson structure on $\mathfrak{h} \oplus V \oplus \mathbb{R}$

Associated Poisson Manifold

- $F_c = \{(\rho, u, f) \in \mathfrak{h} \oplus V \oplus \mathbb{R} : f + (\rho, \rho) = c\} \subset \mathfrak{h} \oplus V \oplus \mathbb{R}$
- The restriction $A_c = A|_{F_c}$ is the cotangent Lie algebroid of a Poisson manifold.
- All of these Poisson structures may be put together into an integrable Poisson structure on $\mathfrak{h} \oplus V \oplus \mathbb{R}$
- $\underbrace{0 \longrightarrow \mathbb{L} \longrightarrow T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}) \longrightarrow A \longrightarrow 0}_{\text{Central Extension of } A \text{ by a line bundle}}$

Moduli

- Recall that the information about the moduli space is encoded on the foliation on X.

Moduli

- Recall that the information about the moduli space is encoded on the foliation on X.

Leafs of A

The leafs of A in $X = \mathfrak{h} \oplus V \oplus \mathbb{R}$ coincide with the symplectic leafs of $T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})$.

Symmetries

- Recall that the information about the symmetries is encoded in the isotropy Lie algebras of $A \to X$.

Symmetries

- Recall that the information about the symmetries is encoded in the isotropy Lie algebras of $A \to X$.

Isotropy Lie algebras

Let $\mathfrak{s}_{\lambda_0}$ be the isotropy Lie algebra of A at $\lambda_0 = (\rho, u, f) \in \mathfrak{h} \oplus V \oplus \mathbb{R}$, and $\mathfrak{g}_{\lambda_0}$ the isotropy Lie algebra of $T^*(\mathfrak{h} \oplus V \oplus \mathbb{R})$. Then

$$0 \longrightarrow \mathbb{R}\lambda_0 \longrightarrow \mathfrak{g}_{\lambda_0} \longrightarrow \mathfrak{s}_{\lambda_0} \longrightarrow 0$$

is an extension of Lie algebras. In particular,

$$\dim \mathfrak{s}_{\lambda_0} = \dim \mathfrak{g}_{\lambda_0} - 1.$$

Construction of Examples

"Integrating" the extension

$$0 \longrightarrow \mathbb{L} \longrightarrow T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}) \longrightarrow A \longrightarrow 0,$$

we obtain:

Construction of Examples

"Integrating" the extension

$$0 \longrightarrow \mathbb{L} \longrightarrow T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}) \longrightarrow A \longrightarrow 0,$$

we obtain:

Theorem (Cahen and Schwachhöfer)

If $\mathrm{s}^{-1}(\lambda)/\exp(\mathbb{R}\lambda)$ is a smooth manifold, then each of its points has a neighborhood which can be embedded in the total space $\mathcal{B}_H(M)$ of an H-structure corresponding to a special symplectic manifold.

Construction of Examples

"Integrating" the extension

$$0 \longrightarrow \mathbb{L} \longrightarrow T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}) \longrightarrow A \longrightarrow 0,$$

we obtain:

Theorem (Cahen and Schwachhöfer)

If $\mathrm{s}^{-1}(\lambda)/\exp(\mathbb{R}\lambda)$ is a smooth manifold, then each of its points has a neighborhood which can be embedded in the total space $\mathcal{B}_H(M)$ of an H-structure corresponding to a special symplectic manifold.
Moreover, if

$$M_\lambda = \frac{(\mathrm{s}^{-1}(\lambda)/\exp(\mathbb{R}\lambda))}{H}$$

is a smooth manifold, then it is a special symplectic manifold.

Thank You!!!