Minimal five dimensional supergravities and complex geometries

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Gauntlett, Gutowski, Hull, Pakis and Reall '03

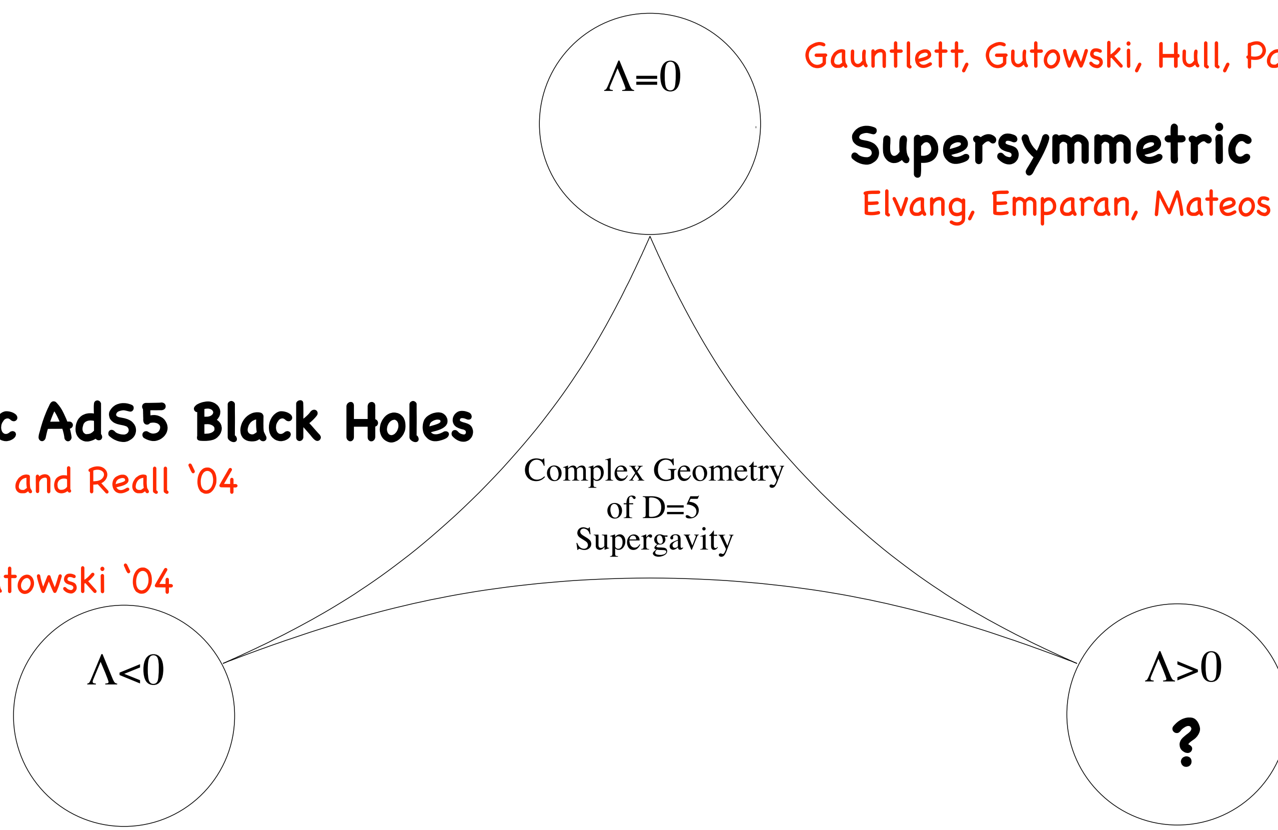
Supersymmetric black ring

Elvang, Emparan, Mateos and Reall '03

Supersymmetric AdS5 Black Holes

Gutowski and Reall '04

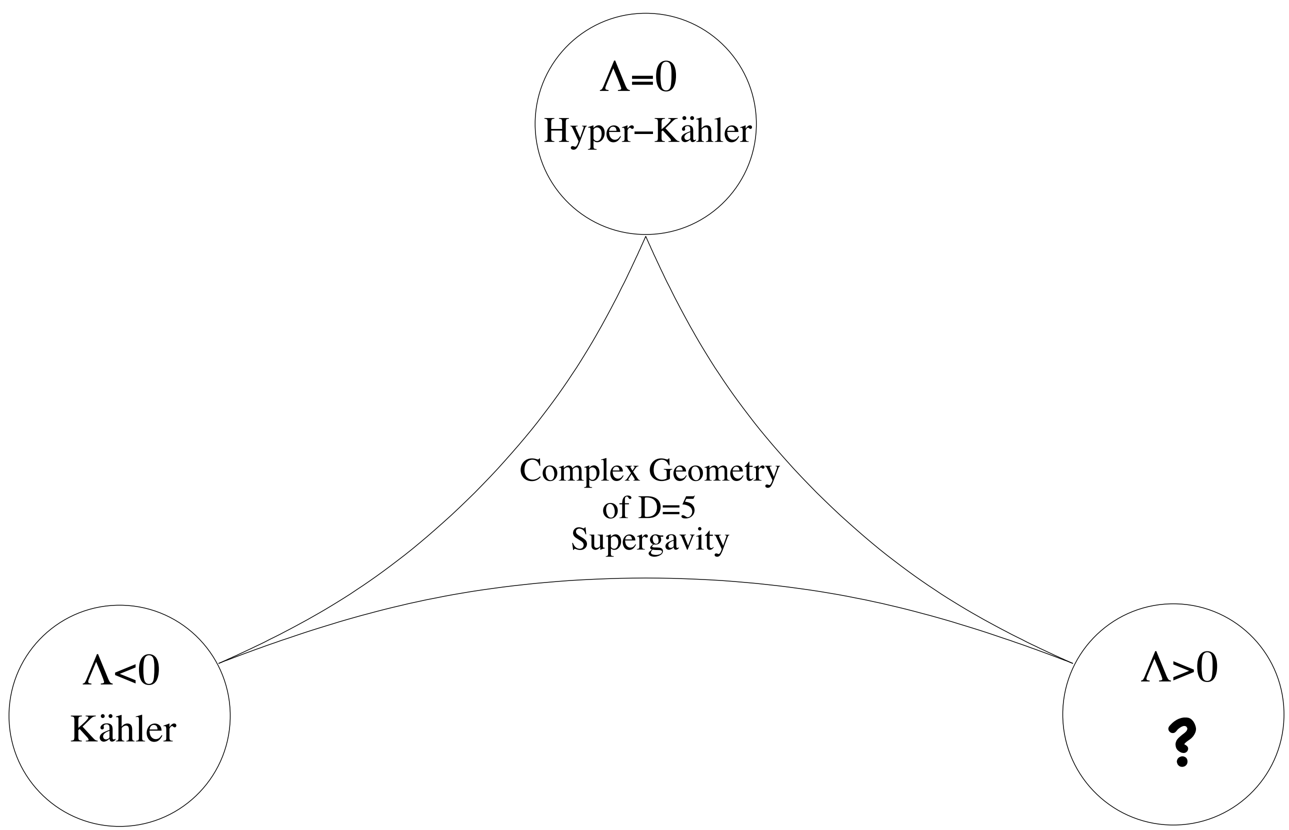
Gauntlett and Gutowski '04



Main questions:

1) Is there any interesting relation with complex geometry in five dimensional minimal de Sitter "supergravity"?

2) Can we use the resulting structure to find interesting solutions?



Outline:

1) Supersymmetry and Complex Geometry

2) Supersymmetry and Black Holes

3) D=5 Minimal Ungauged Supergravity

4) D=5 Minimal Gauged Supergravity

5) D=5 Minimal de Sitter Supergravity
(fake supersymmetry)

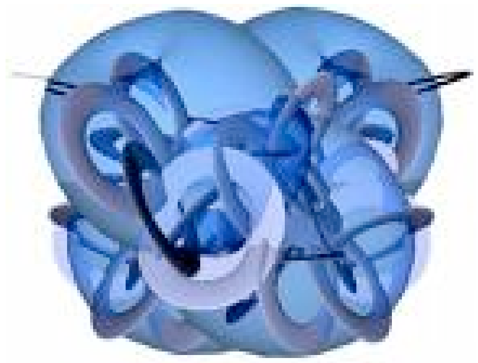
6) Null case

7) Final Remarks

1) Supersymmetry and Complex Geometry

Superstring compactifications

D=10



Calabi-Yau 3-fold

Candelas, Horowitz, Strominger, Witten '85

D=4, N=1 Supersymmetry

Two dimensional non-linear sigma models:

$$\mathcal{L} = -g_{ij}(\phi^i)\partial_\mu\phi^i\partial^\mu\phi^j$$

an N=2 susy extension requires the target space metric to be Kahler

Zumino '79

an N=4 susy extension requires the target space metric to be hyper-Kahler

Alvarez-Gaume, Freedmann '81

$$\mathcal{L} = -g_{ij}(\phi^i)\partial_\mu\phi^i\partial^\mu\phi^j$$
$$+\epsilon_{\mu\nu}e_{ij}\partial_\mu\phi^i\partial_\nu\phi^j$$

Wess-Zumino-Witten couplings are interepreted as torsion potentials leading to Kahler and Hyper Kahler torsion (HKT) geometries

Braaten, Curtright, Zachos '84,'85

HKT geometry also arises in the context of moduli space metrics of five dimensional electrically charged black holes Gibbons, Papadopoulos, Stelle, '97; Gutowski, Papadopoulos '99

2) Supersymmetry and Black Holes

Supersymmetric Black Holes are interesting gravitational objects:

- They are classically and semi-classically stable;
- In many cases there is a no-force condition, which allows for a multi-object solution;

Early example: Take D=4, N=2 supergravity

$$\mathcal{L} = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left(R - \frac{F^2}{4} \right)$$

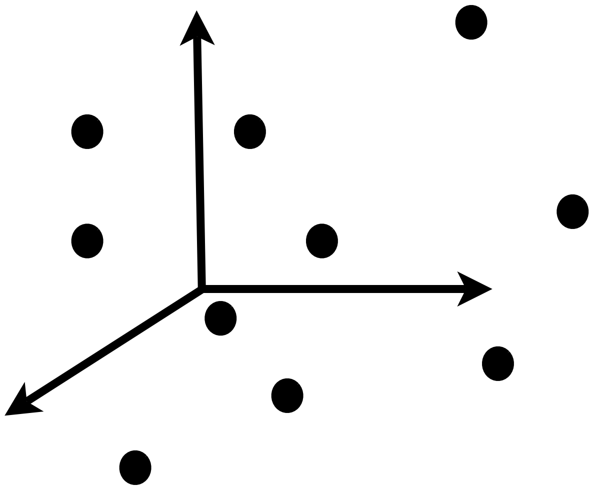
and consider the Majumdar-Papapetrou solutions

Majumdar '47; Papapetrou '47; Hartle and Hawking '72

$$ds^2 = -\frac{dt^2}{H(x)^2} + H(x)^2 ds^2_{\mathbb{E}^3} \ , \qquad A = \frac{dt}{H(x)} \ , \qquad \Delta_{\mathbb{E}^3} H(x) = 0 \ .$$

Multi centre solution

$$H(x) = 1 + \sum_{n=1}^{N} \frac{M_n}{|\mathbf{x} - \mathbf{x_n}|}$$



Balance between electrostatic repulsion and gravitational attraction

Easily extended to D>4; but smoothness of horizon becomes worse:
there is a metric extension which is
analytic (D=4),
C^2 but not C^3(D=5),
C^1 but not C^2(D $\geq$ 6).

Welch '95; Candlish and Reall '07

This exact linearisation is most simply realised by considering the bosonic sector of N=2, D=4 supergravity. The Majumdar-Papapetrou are the most general static solutions admitting Killing spinors and a timelike Killing vector field: Gibbons and Hull '82

$$D\epsilon - \frac{1}{4}F_{ab}\Gamma^{ab}\Gamma\epsilon = 0$$

More generically Tod '83 analysed the Killing spinor equation and showed that all susy solutions with a timelike Killing vector field fall into the class of Israel-Wilson-Perjés metrics Israel and Wilson '72, Perjés '71

$$ds^2 = -|H|^2(dt+\omega_i dx^i)^2 + |H|^2 ds^2_{\mathbb{E}^3} \ , \qquad \nabla \times \vec{\omega} = i(H\nabla H^* - H^*\nabla H)$$

Complexifying H(x) in different ways leads to multi (charged) Taub-NUT or multi Kerr-Newman with Q=M;

Note 1: Force balance for the IWP metrics is more involved, since there are magnetic effects for both gravity and electromagnetism Kastor and Traschen '98.

Note 2: Susy black holes with angular momentum in the non-minimal theory have been found as dimensional reduction of black rings Elvang, Emparan, Mateos and Reall '05. These solutions require multi objects.

3) D=5 Minimal Ungauged Supergravity

$$\mathcal{S} = \frac{1}{16\pi G_5}\int d^5x \left(\sqrt{-g}(R - F^2) - \frac{2}{3\sqrt{3}} A\wedge F\wedge F\right)$$

In five dimensions there are asymptotically flat, regular, susy, rotating black holes, with a connected even horizon: the BMPV black hole: Breckenridge, Myers, Peet and Vafa '96

$$ds^2 = -\frac{(dt+\omega)^2}{H(x)^2} + H(x) ds^2_{\mathbb{E}^4} \ , \qquad A = \frac{\sqrt{3}}{2}\frac{dt+\omega}{H(x)}$$

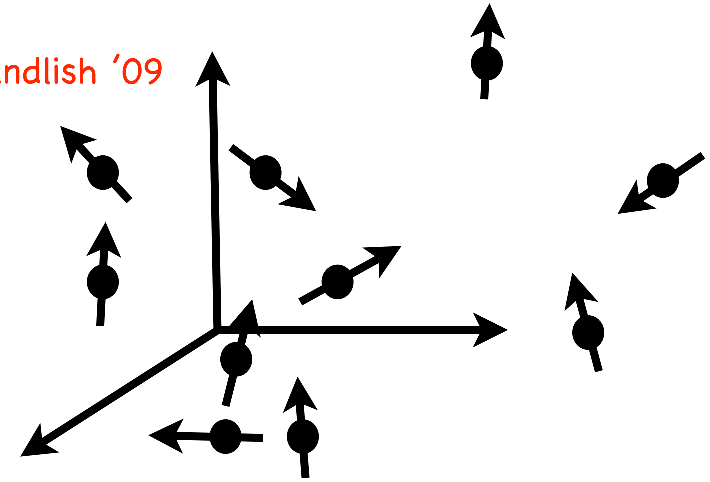
where Gauntlett, Myers and Townsend '98 $\quad \Delta_{\mathbb{E}^4} H(x) = 0 \ , \qquad d\omega = -\star^{(4)} d\omega \ .$

Multi centre solution

$$H(x) = 1 + \sum_{n=1}^{N} \frac{M_n}{|\mathbf{x} - \mathbf{x}_n|^2}$$

$$\omega = dx^j J_j{}^i \partial_i \left(\sum_{n=1}^{N} \frac{J_n}{|\mathbf{x}-\mathbf{x}_n|^2}\right)$$

Not smooth! Candlish '09



More generically Gauntlett, Gutowski, Hull, Pakis and Reall '03 analysed the Killing spinor equation and showed that all susy solutions with a timelike Killing vector field are of the form

$$ds^2 = f^2(dt+\omega)^2 + f^{-1}ds^2_{\mathcal{M}} \ , \qquad F = \frac{\sqrt{3}}{2}d\left(f(dt+\omega)\right) - \frac{G^+}{\sqrt{3}}$$

Method:

1) Choose $\mathcal{M}$ to be hyper-Kahler

2) Decompose: $$f\,d\omega = G^+ + G^-$$

3) Solve: $$dG^+ = 0 \ , \qquad \Delta f^{-1} = \frac{2}{9}(G^+)^2$$

Solutions with $G^+ = 0$ include (multi-BMPV), maximally supersymmetric Godel type universes and black holes in Godel universes Herdeiro '03 ;

Solutions with $G^+ \neq 0$ include supersymmetric black rings.

4) D=5 Minimal Gauged Supergravity

Generical analysis of the Killing spinor equation Gauntlett and Gutowski '04 showed that all susy solutions with a timelike Killing vector field are of the form

$$ds^2 = f^2(dt+\omega)^2 + f^{-1} ds^2_{\mathcal{M}} \; , \qquad F = \frac{\sqrt{3}}{2} d\left(f(dt+\omega)\right) - \frac{G^+}{\sqrt{3}} + \frac{\sqrt{3} g J}{f}$$

Method:

1) Choose $\mathcal{M}$ to be Kahler

2) Compute

$$f = -\frac{24 g^2}{R}$$

$$G^+ = -\frac{1}{2g}\left(\mathcal{R} + \frac{R}{4} J\right)$$

$$\Delta f^{-1} = \frac{2}{9}(G^+)^2 - \frac{g}{f}(G^-)^{mn} J_{mn} - 8\frac{g^2}{f^2}$$

3) Solve constraint: $\quad f d\omega = G^+ + G^-$

In Anti-de-Sitter space supersymmetric black holes <u>must</u> rotate:

In D=3 only the extreme BTZ black hole preserves susy (half) which is enhanced in the zero mass limit. Coussaert and Hanneaux '93

In D=4 the Kerr-Newman-AdS family was found by Carter '68 and the extremal limity analysed by Kostelecky and Perry '95. The zero angular momentum limit of this family is empty AdS4.

In D=5 (most interesting from the viewpoint of AdS/CFT), these black holes were found by Gutowski and Reall '04 (one angular momentum) and Chong, Cvetic, Lu and Pope '04 (two angular momenta).

Note 1: Not every Kahler base space originates a five dimensional solution Figueras, Herdeiro and Paccetti '06

Note 2: No solutions with multi (regular) black holes are known. Do they exist?

Note 3: The CFT description of these black holes has not been completely clarified.

5) D=5 Minimal de Sitter Supergravity (fake supersymmetry)

de Sitter superalgebras have only non-trivial representations in a positive-definite Hilbert space in two dimensions

Pilch, van Bieuwenhuizen and Sohnius; Lukierski and Bowicki '85

We take perspective of fake supesymmetry (analogy to Domain Wall/Cosmology correspondence Skenderis and Townsend '06)

There is a special class of solutions in a gravitational theory with a positive cosmological constant admitting "pseudo-Killing spinors"; fake supersymmetry becomes a solution generating technique.

Relation to type IIB* theory Hull '98; Liu, Sabra and Wen '03

Action and (pseudo) Killing spinor equation:

$$\mathcal{S} = \frac{1}{4\pi G}\int\left(\frac{1}{4}({}^5R - \chi^2)\star 1 - \frac{1}{2}F\wedge\star F - \frac{2}{3\sqrt{3}}F\wedge F\wedge A\right)$$

$$\Big[\partial_M + \frac{1}{4}\Omega_{M,}{}^{N_1N_2}\Gamma_{N_1N_2} - \frac{i}{4\sqrt{3}}F^{N_1N_2}\Gamma_M\Gamma_{N_1N_2} + \frac{3i}{2\sqrt{3}}F_M{}^N\Gamma_N$$
$$+\chi(\frac{i}{4\sqrt{3}}\Gamma_M - \frac{1}{2}A_M)\Big]\epsilon = 0\ ,$$

If a non-trivial solution of the (pseudo) Killing spinor equation exists and the gauge field equations are satisfied, the integrability conditions of the former place constraints on the Ricci tensor.

For the solutions we consider here, in which the Killing spinor generates a timelike vector field these constraints are equivalent to the Einstein equations. This would not be so for the null case.

Computation: assume the existence of at least one non-trivial (pseudo) Killing spinor; place constraints on the spin connections and gauge field.

Use spinorial geometry techniques:
one takes the space of Dirac spinors to be the space of complexified forms on $\mathbb{R}^2$, which is spanned over $\mathbb{C}$ by

$$\{1, e_1, e_2, e_{12}\} \;, \qquad e_{12} = e_1 \wedge e_2$$

Action of complexified gamma matrices on these spinors is given by

$$\Gamma_\alpha = \sqrt{2} e_\alpha \wedge \;, \qquad \Gamma_{\bar{\alpha}} = \sqrt{2} i_{e^\alpha} \;, \qquad \alpha = 1, 2$$

$$\Gamma_0 1 = -i1, \quad \Gamma_0 e^{12} = -ie^{12}, \quad \Gamma_0 e^j = ie^j \quad j = 1, 2 \;,$$

Spacetime metric: $ds^2 = -(\mathbf{e}^0)^2 + 2\delta_{\alpha\bar{\beta}} \mathbf{e}^\alpha \mathbf{e}^{\bar{\beta}}$.

Killing spinor

originates a timelike vector

originates a null vector

Canonical form using Spin(4,1)

$\epsilon = f1$

$\epsilon = 1 + e_1$

Define the 1-form: $$V = \mathbf{e}^0$$

Introduce t co-ordinate such that the dual vector field is

$$V = -\frac{\partial}{\partial t}$$

Then, computation leads to:

$$\mathbf{e}^0 = dt + \frac{2\sqrt{3}}{\chi}\mathcal{P} + e^{\frac{\chi}{\sqrt{3}}t}\mathcal{Q}\ , \qquad \mathbf{e}^\alpha = e^{-\frac{\chi}{2\sqrt{3}}t}\hat{\mathbf{e}}^\alpha$$

$$\mathcal{L}_V\hat{\mathbf{e}}^\alpha = 0\ , \quad \mathcal{L}_V\mathcal{Q} = 0\ , \quad \mathcal{L}_V\mathcal{P} = 0\ .$$

We refer to the 4-manifold with t-independent metric

$$ds^2_B = 2\delta_{\alpha\bar{\beta}}\hat{\mathbf{e}}^\alpha\hat{\mathbf{e}}^{\bar{\beta}}$$

as the "base space" B.

Part of the geometrical constraints imposed by the Killing spinor equation are equivalent to:

$$dJ^i = -2\mathcal{P} \wedge J^i \,, \qquad i = 1, 2, 3 \,,$$

where:

$$\begin{aligned} J^1 &= \hat{\mathbf{e}}^1 \wedge \hat{\mathbf{e}}^2 + \hat{\mathbf{e}}^{\bar{1}} \wedge \hat{\mathbf{e}}^{\bar{2}} \,, \\ J^2 &= i\hat{\mathbf{e}}^1 \wedge \hat{\mathbf{e}}^{\bar{1}} + i\hat{\mathbf{e}}^2 \wedge \hat{\mathbf{e}}^{\bar{2}} \,, \\ J^3 &= -i\hat{\mathbf{e}}^1 \wedge \hat{\mathbf{e}}^2 + i\hat{\mathbf{e}}^{\bar{1}} \wedge \hat{\mathbf{e}}^{\bar{2}} \,, \end{aligned}$$

defines a triplet of anti-self-dual almost complex structures on B which satisfy the algebra of the imaginary unit quaternions.

Thus, B is hyper-Kahler with torsion, HKT: $\nabla^+ J^i = 0 \,,$

$$\Gamma^{(+)i}{}_{jk} = \left\{{}^{\,i}_{jk}\right\} + H^i{}_{jk} \,, \qquad H = \star_4 \mathcal{P} \,.$$

Note: Without loss of generality we can take B to be strong HKT, i.e. dH=0

Method:

1) Take the base space B to be a four dimensional HKT geometry with torsion tensor H.

2) The one form $\mathcal{P}$ is given by $\mathcal{P} = - \star_4 H$

3) Choose a 1-form $\mathcal{Q}$ obeying the constraints:

$$\left(d\mathcal{Q} - 2\mathcal{P} \wedge \mathcal{Q}\right)^+ = 0 \ , \qquad d \star_4 \mathcal{Q} + \frac{16}{\sqrt{3}\chi^3} d\mathcal{P} \wedge d\mathcal{P} = 0 \ .$$

4) The solution is given by:

$$ds^2 = -\left(dt + \frac{2\sqrt{3}}{\chi}\mathcal{P} + e^{\frac{\chi}{\sqrt{3}}t}\mathcal{Q}\right)^2 + e^{-\frac{\chi}{\sqrt{3}}t} ds_B^2 \ , \quad A = \frac{1}{2\sqrt{3}}dt + \frac{1}{\chi}\mathcal{P} + \frac{\sqrt{3}}{2}e^{\frac{\chi}{\sqrt{3}}t}\mathcal{Q} \ ,$$

Note: The t-dependence is explicit; for the scalar curvature:

$$\mathcal{R} = \frac{5}{3}\chi^2 + \frac{e^{\frac{2\chi}{\sqrt{3}}t}}{3}\left[\frac{(d\mathcal{P})^2}{\chi^2} - \frac{\chi^2}{2}e^{\frac{\chi}{\sqrt{3}}t}\mathcal{Q}^2 + \frac{3}{4}e^{\frac{2\chi}{\sqrt{3}}t}(d\mathcal{Q} - 2\mathcal{P} \wedge \mathcal{Q})^2\right] \ ,$$

Regularity at $t = \pm\infty$ requires: $\mathcal{Q} = 0$, $d\mathcal{P} = 0$, and B is conformally hyper-Kahler

Examples 1: solutions with conformally hyper-Kahler base

In this case: $$d\mathcal{P} = 0$$

After some coordinate transformations the solution can be cast in the form:

$$ds^2 = -f^2(dt+a)^2 + f^{-1}ds^2_{HK} \;, \qquad F = \frac{\sqrt{3}}{2} d\Big(f(dt+a)\Big) \;,$$

where: $$f^{-1} = V - \frac{\chi}{\sqrt{3}}t \;,$$

and the constraints become:

$$\Delta_{HK}V = 0 \;, \qquad (da)^+ = 0 \;.$$

exactly the form of the solutions of minimal ungauged SUGRA with $G^+ = 0$, except for the linear t-dependent term.

Note: de Sitter space is obtained by taking $HK = \mathbb{R}^4 \;, V = const. \;, a = 0 \;.$

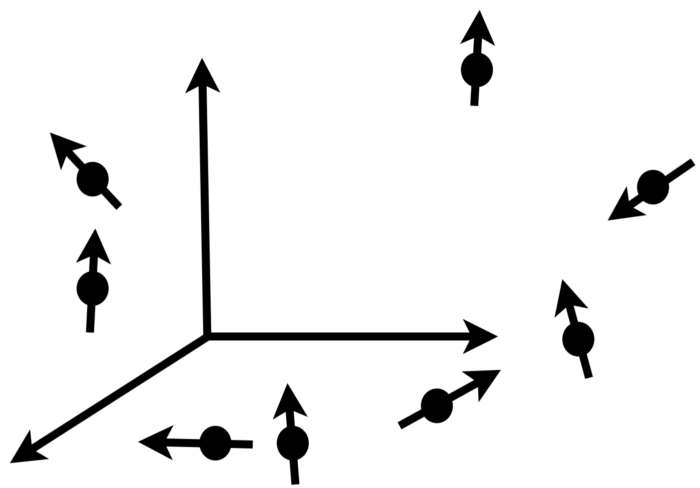
Theorem: Any solution of D=5 minimal de Sitter SUGRA with a supercovariantly constant spinor and a base space which is conformal to a hyper-Kahler manifold can be obtained from a "seed" solution of the minimal ungauged SUGRA theory simply by adding a linear time dependence in the harmonic function.

Generalises earlier result by Behrnt and Cvetic '03

Makes clear why we can superimpose certain solutions (like the BMPV black hole Klemm and Sabra '01 or Godel type universes Behrnt and Klemm '04) with a positive cosmological constant

Suggests that solutions with $G^+ \neq 0$ do not generalise easily to de Sitter space; most notably the black ring

In contrast with the AdS theory it admits multi-black hole solutions:



Multi BMPV:

$$f^{-1} = 1 + \sum_{n=1}^{N} \frac{\mu_n}{|\mathbf{x} - \mathbf{x}_n|^2}$$

$$a = dx^j J_j{}^i \partial_i \left(\sum_{n=1}^{N} \frac{J_n}{|\mathbf{x} - \mathbf{x}_n|^2} \right)$$

Five dimensional, rotating, generalisation of the Kastor-Traschen '93 solutions;

Examples 2: solutions with a tri-holomorphic Killing vector

$$\mathcal{L}_X h_B = 0, \qquad \mathcal{L}_X J^i = 0, \qquad i = 1, 2, 3 \ ,$$

HKT manifolds with a tri-holomorphic Killing vector field have been classified Chave, Tod, Valent '96; Gauduchon and Tod '98

Their structure is specified in terms of a constrained 3-dimensional Einstein-Weyl geometry:

$$(\gamma_{ij}, u_i, u_0)$$

$$\star_E du = -du_0 - u_0 u \ ,$$

$$^{(E)}R_{ij} + \nabla_{(i}u_{j)} + u_i u_j = \gamma_{ij}(\frac{1}{2}u_0^2 + u_k u^k) \ ,$$

$$d \star_E u = 0 \ .$$

Basic example: round 3-sphere

$$ds_E = b^2 \left(d\theta^2 + \sin^2\theta (d\phi^2 + \sin^2\phi d\psi^2) \right) \ , \qquad u = 0 \ , \qquad u_0 = -\frac{2}{b} \ ,$$

Simplest solutions of constraints leads to singular universes.

6) Null case:

Killing spinor

originates a
timelike vector

originates a
null vector

$\epsilon = f1$

Canonical form
using Spin(4,1)

$\epsilon = 1 + e_1$

General form of the solution:

$$ds^2 = 2du\left(dv + \left(H - \frac{\chi^2}{8}v^2\right)du + \chi v\mathcal{B} + \phi\right) - ds^2_{\rm GT}\,, \qquad F = \frac{\chi}{4}du \wedge dv + d\mathcal{B}\,,$$

Method:

1) Choose a Gaudochon-Tod space as base $ds^2_{\rm GT} = \delta_{ij}\mathbf{E}^i\mathbf{E}^j$

$$\tilde{d}\mathbf{E}^i = -\frac{\sqrt{3}\chi}{2}\star_3\mathbf{E}^i + \chi\mathcal{B}\wedge\mathbf{E}^i\,,$$

2) Choose a 1-form on the base obeying

$$\frac{\chi}{4}\phi - \dot{\mathcal{B}} - \frac{1}{2\sqrt{3}}\star_3\left(\tilde{d}\phi + \chi\mathcal{B}\wedge\phi - \mathbf{E}^i\wedge\dot{\mathbf{E}}^i\right) = 0$$

3) Choose a function H obeying

$$\Box_3 H + \chi\mathcal{B}\cdot\tilde{d}H = \tilde{\nabla}^i\dot{\phi}_i + (\ddot{\mathbf{E}}^i)_i + \chi\phi\cdot\dot{\mathcal{B}} - 4\dot{\mathcal{B}}^2 - 2\sqrt{3}\star_3\left(\frac{\chi}{4}\phi - \dot{\mathcal{B}}\right)_{ij}(\dot{\mathbf{E}}^i)_j$$

None of these quantities depends on v

Generically a Kundt geometry, as in the Minkowski and AdS cases; But, unlike these cases, the null vector field is not Killing.

For $\mathcal{B} = 0$ there are nice properties:

- The null vector field becomes recurrent:

$$\nabla_\mu N^\nu = -\frac{1}{4}\chi^2 v N_\mu N^\nu$$

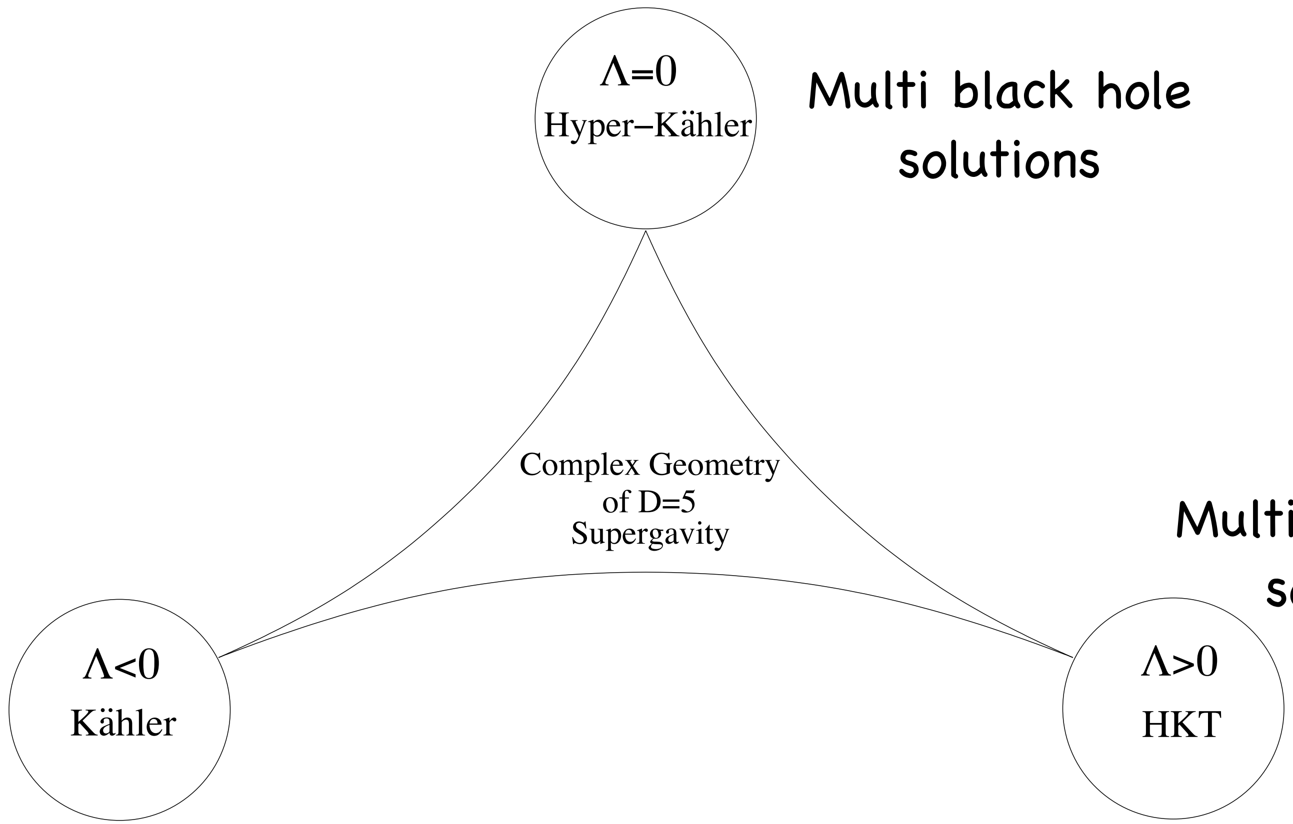
This implies that the holonomy is contained in Sim(3)

- All scalar invariants constructed from the curvature are constant and all invariants with covariant derivatives of the curvature are zero.

7) Final remarks:

Nice geometrical picture!

Multi black hole solutions



Multi black hole solutions

If B is conformally hyper-Kahler, nice connection with the minimal ungauged theory

If B is not conformally hyper-Kahler, can one find more interesting solutions? In particular rings?