On the Cohomology of Almost Complex Manifolds

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 $\mathcal{K}_J^t(M) = \mathcal{K}_J^c(M) + \left[(H^{2,0}_{\overline{\partial}}(M) \oplus H^{0,2}_{\overline{\partial}}(M))_{\mathbb{R}} \cap H^2(M,\mathbb{R}) \right],$



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• Extend the previous decomposition theorem to the non integrable almost complex structures.

Li and Zhang introduce the coohomology groups

 $H^{1,1}_J(M)_{\mathbb{R}} = \{ [\alpha] \mid \alpha \in \mathcal{Z}^{1,1}_J \},\$

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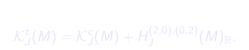
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For a finite set S of pairs of integers, let

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Then there is a natural inclusion



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$$H^{2}(M,\mathbb{R}) = H^{1,1}_{J}(M)_{\mathbb{R}} \oplus H^{(2,0),(0,2)}_{J}(M)_{\mathbb{R}}.$$

• J is \mathcal{C}^{∞} pure if and only if

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Definition (Li, Zhang 2008) A smooth almost complex structure J on M is said to be C^{∞} pure and full if

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Theorem (Draghici, Li, Zhang, 2008, to appear in *Inter. Math. Res. Not.*) If M is a compact 4-dimensional manifold, then every almost complex structure J on M is C^{∞} pure and full. Thus, there is a direct sum decomposition

$$H^2(M,\mathbb{R}) = H^+_J(M) \oplus H^-_J(M)$$
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Definition (Li, Zhang 2008) An almost complex structure J is said to be

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$$H^J_{1,1}(M)_{\mathbb{R}} \cap H^J_{(2,0),(0,2)}(M)_{\mathbb{R}} = \{0\},\$$

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Let (M, ω) be an almost symplectic 2n-dimensional compact manifold and J be a C^{∞} -pure and full almost complex structure calibrated by ω .

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Proof. We start to prove that J is pure.

We show that

 $\pi_{1,1}\mathcal{B}_2\cap \mathcal{Z}_{1,1}^J=\mathcal{B}_{1,1}^J.$

Since $\pi_{1,1}\mathcal{B}_2 \cap \mathcal{Z}_{1,1}^J \supset \mathcal{B}_{1,1}^J$, we need to prove the other inclusion. Let $T \in \pi_{1,1}\mathcal{B}_2 \cap \mathcal{Z}_{1,1}^J$; then $T = \pi_{1,1}dS$, where S is a real 3-current and $d(\pi_{1,1}dS) = 0$. We have to show that $T = \pi_{1,1}dS$ is a boundary, i.e. that it vanishes on any closed real 2-form α .

• If α is exact, then $(\pi_{1,1}dS)(\alpha) = 0$. Suppose that $[\alpha] \neq 0 \in H^2(M, \mathbb{R})$, then since J is \mathcal{C}^{∞} pure and full, we can write

 $\alpha = \alpha_1 + \alpha_2 + \boldsymbol{d}\gamma,$

with $\alpha_1 \in \mathcal{Z}_J^{1,1}$, $\alpha_2 \in \mathcal{Z}_J^{(2,0),(0,2)}$ and $\gamma \in \Omega^1(M)_{\mathbb{R}}$. Therefore,

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If n = 2, to prove that J is also full, we have to show that

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holds.

Let $[T] \in H_2(M, \mathbb{R})$; then there exists a smooth closed 2-form β on M such that $[T] = [\beta]$.

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• The assumption in the last Theorem that any (1,1) (respectively (2,0) + (0,2)) de Rham class contains a harmonic representative seems quite strong.

• There are examples of compact non-Kähler solvmanifolds satisfying the above assumption.

• To get the pureness of J, it is enough to assume that J is C^{∞} full (see also Li, Zhang).

• By Draghici, Li, Zhang, if n = 2, then any almost complex structure J is C^{∞} pure and full

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• *G* simply-connected nilpotent Lie group whose Lie algebra has structure equation

$$\left\{ egin{array}{ll} de^j=0, & j=1,\ldots,4, \ de^5=e^1\wedge e^2, \ de^6=e^1\wedge e^3, \end{array}
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 $M = \Gamma \setminus G$ (compact) nilmanifold.

• Left-invariant almost complex structure on *M*, defined by the (1,0)-forms

$$\eta^1 = e^1 + i e^2, \quad \eta^2 = e^3 + i e^4, \quad \eta^3 = e^5 + i e^6,$$

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Proof n = 2 the result follows by the last Theorem.
n > 2, J is pure. We have to show

 $H_2(M,\mathbb{R}) = H_{1,1}^J(M)_{\mathbb{R}} + H_{(2,0),(0,2)}^J(M)_{\mathbb{R}}.$

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• $\mathfrak{s} = \mathfrak{sol}_3 \oplus \mathfrak{sol}_3$ 6-dimensional completely solvable Lie algebra with structure equations

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• The corresponding simply connected Lie group S has a compact quotient $M^6 = \Gamma \setminus S$.

 $H^{2}(M^{6},\mathbb{R}) = \mathbb{R} < [f^{14}], [f^{25}], [f^{36}] > 1$



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 $\varphi^1 = f^1 + if^4, \ \varphi^2 = f^2 + if^5, \ \varphi^3 = f^3 + if^6.$

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Consequently,

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$$\begin{split} \varphi_t^1 &= f^1 + i \left(\frac{2t}{(1-t^2)} f^1 + \frac{1+t^2}{1-t^2} f^4 \right) \,, \\ \varphi_t^2 &= f^2 + i \left(\frac{2t}{(1-t^2)} f^2 + \frac{1+t^2}{1-t^2} f^5 \right) \,, \\ \varphi_t^3 &= f^3 + i \left(\frac{2t}{(1-t^2)} f^3 + \frac{1+t^2}{1-t^2} f^6 \right) \,. \end{split}$$



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• J_t is a family of \mathcal{C}^∞ pure and full almost complex structures, in fact pure and full, since

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