# On the Cohomology of Almost Complex Manifolds

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# Introduction

- $(M, J)$ $2n$-dimensional compact almost complex manifold.

The *J-tamed symplectic cone* is

$$\mathcal{K}_J^t(M) = \{[\omega] \in H^2(M, \mathbb{R}) \,|\, \omega \text{ is symplectic and tamed by } J\}$$

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- Extend the previous decomposition theorem to the non integrable almost complex structures.

Li and Zhang introduce the coohomology groups

$$H_J^{1,1}(M)_{\mathbb{R}} = \{[\alpha] \,|\, \alpha \in \mathcal{Z}_J^{1,1}\},$$

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- $J$ almost complex structure on a compact $2n$-dimensional manifold $M$.

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$$H_J^S(M)_{\mathbb{R}} = \left\{ [\alpha] \;\middle|\; \alpha \in \mathcal{Z}_J^S \right\} = \frac{\mathcal{Z}_J^S}{\mathcal{B}}.$$

Then there is a natural inclusion

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**Definition** (Li, Zhang 2008) A smooth almost complex structure $J$ on $M$ is said to be $\mathcal{C}^\infty$ *pure and full* if

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# Pure and full almost complex structures

- $(M, J)$ compact almost complex manifold of real dimension $2n$.
- $\mathcal{E}_k(M)$ $k$-currents on $M$, i.e. the topological dual of $\Omega^{2n-k}(M)$.

$k$-forms are $(2n-k)$-currents $\implies$ $k$-th de Rham homology group

$$H_k(M, \mathbb{R}) \cong H^{2n-k}(M, \mathbb{R})$$

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Set (Li-Zhang)

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where $\mathcal{B}$ denotes the space of currents which are boundaries.

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# Pure and full almost complex structures

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If $n = 2$, to prove that $J$ is also full, we have to show that

$$H_2(M,\mathbb{R}) = H^J_{1,1}(M)_{\mathbb{R}} + H^J_{(2,0),(0,2)}(M)_{\mathbb{R}} \,.$$

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# Pure and full almost complex structures

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# Example

- $G$ simply-connected nilpotent Lie group whose Lie algebra has structure equation

$$\begin{cases} de^j = 0, & j = 1, \dots, 4, \\ de^5 = e^1 \wedge e^2, \\ de^6 = e^1 \wedge e^3, \end{cases}$$

$M = \Gamma \backslash G$ (compact) nilmanifold.

- Left-invariant almost complex structure on $M$, defined by the $(1,0)$-forms

$$\eta^1 = e^1 + ie^2, \quad \eta^2 = e^3 + ie^4, \quad \eta^3 = e^5 + ie^6,$$

is not $C^\infty$-pure, since

$$[\mathrm{Re}(\eta^1 \wedge \overline{\eta}^2)] = [e^{13} + e^{24}] = [e^{24}] = [\mathrm{Re}(\eta^1 \wedge \eta^2)] = [e^{13} - e^{24}].$$

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# Hard Lefschetz condition

- $(M, \omega)$ $2n$-dimensional compact symplectic manifold
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$$\omega^k : \Omega^{n-k}(M) \to \Omega^{n+k}(M)\,,\ \alpha \mapsto \omega^k \wedge \alpha$$

induce an isomorphism in cohomology.

**Theorem** (Fino,—, to appear in *J. Geom. Anal.*)
*Let $(M, \omega)$ be a $2n$-dimensional compact symplectic manifold which satisfies the Hard Lefschetz condition. Let $J$ be a $C^\infty$ pure and full almost complex structure calibrated by $\omega$. Then $J$ is pure and full.*

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**Proof** $n = 2$ the result follows by the last Theorem.

- $n > 2$, $J$ is pure. We have to show

$$H_2(M,\mathbb{R}) = H^J_{1,1}(M)_{\mathbb{R}} + H^J_{(2,0),(0,2)}(M)_{\mathbb{R}}\,.$$

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# Families of $\mathcal{C}^\infty$ pure and full almost complex structures

- $\mathfrak{s} = \mathfrak{sol}_3 \oplus \mathfrak{sol}_3$ 6-dimensional completely solvable Lie algebra with structure equations

$$\begin{cases} df^1 = 0\,, \\ df^2 = -f^1 \wedge f^2\,, \\ df^3 = f^3 \wedge f^4\,, \\ df^4 = 0, \\ df^5 = f^1 \wedge f^5\,, \\ df^6 = f^4 \wedge f^6\,. \end{cases}$$

- The corresponding simply connected Lie group $S$ has a compact quotient $M^6 = \Gamma \backslash S$.

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$J_0$ is almost-Kähler with respect to

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- By M. Fernández, V. Muñoz, J. A. Santisteban (2003) *Int. J. Math. Math. Sci.*, $(M^6, J_0, \omega)$ satisfies the Hard Lefschetz condition and

$$H^2(M^6, \mathbb{R}) = H^{1,1}_{J_0}(M^6)_{\mathbb{R}}$$

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Consequently,

$$J_t = \begin{pmatrix} \frac{2t}{1-t^2} I & -\frac{1+t^2}{1-t^2} I \\ \frac{1+t^2}{1-t^2} I & -\frac{2t}{1-t^2} I \end{pmatrix}$$

and a basis of $(1,0)$-forms for $J_t$ is

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