Global controllability for Burgers equation

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Introduction
Case of Navier-Stokes Equations
(\bar{y}, \bar{p}) : “ideal” solution of Navier-Stokes equations (for example a stationnary solution).

\[
\begin{align*}
\frac{\partial \bar{y}}{\partial t} - \nu \Delta \bar{y} + \bar{y} \cdot \nabla \bar{y} + \nabla \bar{p} &= f \text{ in } \Omega \times (0, T), \\
\text{div } \bar{y} &= 0 \text{ in } \Omega \times (0, T), \\
\bar{y} &= 0 \text{ on } \Gamma \times (0, T) \\
\bar{y}(0) &= \bar{y}_0 \text{ in } \Omega.
\end{align*}
\]

Consider a solution of the controlled system, starting from a different initial value

\[
\begin{align*}
\frac{\partial y}{\partial t} - \nu \Delta y + y \cdot \nabla y + \nabla p &= f + v \cdot \mathbb{1}_\omega \text{ in } \Omega \times (0, T), \\
\text{div } y &= 0 \text{ in } \Omega \times (0, T), \\
y &= 0 \text{ on } \Gamma \times (0, T) \\
y(0) &= y_0 \text{ in } \Omega,
\end{align*}
\]

\(\mathbb{1}_\omega : \text{ characteristic function of a (little) subset } \omega \text{ of } \Omega.\)
Exact Controllability to Trajectories:

Can we find a control $v$ such that

$$y(T) = \bar{y}(T) ?$$

i.e. can we reach exactly in finite time the “ideal” trajectory $\bar{y}$?

Local version: same result provided $||y_0 - \bar{y}_0||$ is small enough.

\[ H = \{ y \in L^2(\Omega)^3, \ \text{div} \, y = 0, \ y \cdot \nu = 0 \text{ on } \Gamma \}. \]

**Theorem 1** Let us assume that

\[ \bar{y}_0 \in H \cap L^4(\Omega)^3, \ \bar{y} \in L^\infty(\Omega \times (0, T))^3 \]

and

\[ \frac{\partial \bar{y}}{\partial t} \in L^2(0, T; L^\sigma(\Omega))^3, \ \sigma > \frac{6}{5} \]

then there exists \( \eta > 0 \) such that for every \( y_0 \in H \cap L^4(\Omega)^3 \) such that \( \|y_0 - \bar{y}_0\|_{L^4(\Omega)^3} \leq \eta \), there exists a control \( v \in L^2(0, T; L^2(\omega))^3 \) and a solution \((y, p)\) of (2) such that

\[ y(T) = \bar{y}(T). \]
Among open problems:

Can the result be global (at least to achieve 0)?

Open problem except for control on the whole boundary: combining results of Coron for approximate controllability and a local exact controllability result (Fursikov-Imanuivilov or result mentioned above).

Can we use a more “nonlinear” method?
Case of Burgers Equations
For 1-d Burgers equation: counter-example due to Guerrero-Imanuvilov. Therefore no global exact controllability.
Global exact boundary controllability for the 2-d Burgers equation

\[
\frac{\partial u}{\partial t} - \Delta u + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = f \quad \text{in } Q = (0,T) \times \Omega, \tag{3}
\]

\[
u|_{\Gamma_0} = 0, \quad \nu|_{\Gamma_1} = h, \tag{4}
\]

\[
u(0, \cdot) = u_0, \tag{5}
\]

\[
u(T, \cdot) = 0. \tag{6}
\]

Without loss of generality we may assume that \( \Omega \) is included in the rectangle \( 0 \leq x_2 - x_1 \leq A, \quad -B \leq x_1 + x_2 \leq B \) with \( A \) and \( B \) two positive constants.
Theorem 2 Let us assume that

\[ \Gamma_0 \subset \{ x \in \Gamma \mid x_1 - x_2 = 0 \} \]  

(or \( \Gamma_0 \) is empty which is allowed). Suppose that \( f \in L^2(0,T; L^2(\Omega)) \) and that there exists \( T_0 \in (0,T) \) such that \( f(t,x) = 0, \forall t \geq T_0 \).

Then for every \( u_0 \in L^2(\Omega) \) there exists a solution \( u \in L^2(0,T; H^{1,\Gamma_0}(\Omega)) \cap C([0,T]; L^2(\Omega)) \) such that \( t^2.u \in H^{1,2}(Q) = H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^{1,\Gamma_0}(\Omega)) \) to problem (3)-(5) satisfying (6) (and a corresponding control \( h \)).
Proof: related to the return method by Coron but different. Use of a special solution of Burgers equation that we can drive to zero whenever we want.
First of all some existence and regularity results for Burgers equations (good exercises !!)

**Proposition 3** For every \( f \in L^2(0, T; H^{-1}(\Omega)) \) and \( u_0 \in L^2(\Omega) \) there exists a unique solution \( u \) to 2-D Burgers equation with \( u \in L^2(0, T; H^1_0(\Omega)) \) and we have

\[
\|u\|_{L^2(0,T;H^1_0(\Omega))} + \|u\|_{C([0,T];L^2(\Omega))} \leq C(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;H^{-1}(\Omega))}).
\]

If \( f \in L^2(0, T; L^2(\Omega)) \) and \( u_0 \in H^1_0(\Omega) \) then \( u \in H^{1,2}(Q) = H^1(0, T; L^2(\Omega)) \) and we have

\[
\|u\|_{H^{1,2}(Q)} \leq C(\|u_0\|_{H^1_0(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)}^5 + \|f\|_{L^2(0,T;L^2(\Omega))}^5).
\]
Proposition 4 Let us assume that \( f \in L^2(0, T; L^2(\Omega)) \) and that \( u_0 \in L^2(\Omega) \). Then \( t^2.u \in H^{1,2}(Q) \) which implies that for every \( \eta > 0 \), \( u \in C([\eta, T]; H^1_0(\Omega)) \cap L^2(\eta, T; H^2(\Omega)) \) and \( \frac{\partial u}{\partial t} \in L^2(\eta, T; L^2(\Omega)) \). Moreover we have the following estimate

\[
\|t^2.u\|_{H^{1,2}(Q)} \leq C(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0, T; L^2(\Omega))} + \\
+\|u_0\|^{13}_{L^2(\Omega)} + \|f\|^{13}_{L^2(0, T; L^2(\Omega))}).
\]
On the time interval $(0, T_0)$ set $h(t, x) = 0$ and leave the system evolve without control. For every $\eta > 0$, we have

$$u \in C([\eta, T_0]; H^1_0(\Omega)) \cap L^2(\eta, T_0; H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(\eta, T_0; L^2(\Omega))$$

and we write

$$u(T_0, \cdot) = u_1 \in H^1_0(\Omega) \subset L^p(\Omega), \quad \forall p, \ 1 \leq p < +\infty.$$ 

Now we set

$$\delta_0 = \frac{T - T_0}{4} > 0.$$ 

We will construct a solution $u$ in the interval $(T_0, T_0 + 3\delta_0)$ (and a corresponding control) such that $u(T_0 + 3\delta_0, \cdot)$ is as small as desired in the norm $H^1_0(\Omega)$. 

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First of all we construct a very specific solution $U$ of the 2-d Burgers equation.

Let $w(t, z)$ be a solution to the heat equation

$$\frac{\partial w}{\partial t} - 2 \frac{\partial^2 w}{\partial z^2} = 0 \quad z \in (0, A), \ t > T_0, \quad (9)$$

$$w(t, 0) = 0, \quad w(t, A) = v(t), \quad (10)$$

$$w(T_0, \cdot) = 0, \quad (11)$$

where $v(\cdot)$ is a boundary control which will be determined later on. This control will be chosen regular so that $w$ will also be regular.

We now set

$$U(t, x) = w(t, x_2 - x_1). \quad (12)$$
We have
\[ \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \quad \frac{\partial U^2}{\partial x_1} + \frac{\partial U^2}{\partial x_2} = 0 \]
so that for every \( N > 0 \), \( N.U \) is a regular solution of the 2-d Burgers equation
\[
\frac{\partial (N.U)}{\partial t} - \Delta (N.U) + \frac{\partial (N.U)^2}{\partial x_1} + \frac{\partial (N.U)^2}{\partial x_2} = 0 \quad \text{in} \ (T_0, T) \times \Omega,
\]
\[ N.U|_{\Gamma_0} = 0, \]
\[ N.U(T_0, \cdot) = 0. \]
Notice that the value of $N.U$ on $(T_0, T) \times \Gamma_1$, which will be a boundary control $h$ and which depends on $v$, does not appear explicitly. If $\delta$ is any number such that $0 < \delta \leq \delta_0$, from the controllability results for the heat equation, we can choose this control $h$ (and in fact $v$) on $(T_0 + \delta, T_0 + 2\delta_0)$ such that

$$N.U(T_0 + 2\delta_0, \cdot) = 0.$$ 

On the interval $(T_0, T_0 + 2\delta_0)$ we look for $u$ in the form

$$u = y + N.U,$$

where $N$ is a large parameter to be determined later on and $y$ is chosen to vanish on the whole boundary $\Gamma$. 

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Therefore, $y$ must satisfy the following equation

\[
\frac{\partial y}{\partial t} - \Delta y + 2N.U\left(\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2}\right) + \frac{\partial y^2}{\partial x_1} + \frac{\partial y^2}{\partial x_2} = 0
\]

in $(T_0, T_0 + 2\delta_0) \times \Omega$, \hspace{1cm} (14)

$y|_{\Gamma} = 0$, \hspace{1cm} (15)

$y(T_0, \cdot) = u_1$. \hspace{1cm} (16)

**Lemma 5** There exists a unique solution $y$ to (14), (15), (16) with $y \in C([T_0, T_0 + 2\delta_0]; H^1_0(\Omega)) \cap L^2(T_0, T_0 + 2\delta_0; H^2(\Omega))$, $\frac{\partial y}{\partial t} \in L^2(T_0, T_0 + 2\delta_0; L^2(\Omega))$ and for every $t_0, t_1$ with $T_0 \leq t_0 \leq t_1 \leq T_0 + 2\delta_0$ and every $p \geq 1$ we have

\[
\|y(t_1, \cdot)\|_{L^p(\Omega)} \leq \|y(t_0, \cdot)\|_{L^p(\Omega)}. \hspace{1cm} (17)
\]
Proof.

Existence, uniqueness and regularity of $y$ is classical as (14) is essentially a Burgers equation. To show that the $L^p$-norm of $y$ is decreasing, multiply equation (14) by $|y|^{p-2} y$ with $p \geq 1$. We obtain

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} |y|^p dx + (p - 1) \int_{\Omega} |y|^{p-2} |\nabla y|^2 dx = 0$$

since

$$\int_{\Omega} U.(\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2})|y|^{p-2} y dx = \frac{1}{p} \int_{\Omega} U.(\frac{\partial |y|^p}{\partial x_1} + \frac{\partial |y|^p}{\partial x_2}) dx = 0$$

and

$$\int_{\Omega} (\frac{\partial y^2}{\partial x_1} + \frac{\partial y^2}{\partial x_2})|y|^{p-2} y dx = \frac{2}{p+1} \int_{\Omega} (\frac{\partial |y|^p y}{\partial x_1} + \frac{\partial |y|^p y}{\partial x_2}) dx = 0.$$
Let us now define a function $\beta$ by

$$\beta(x) = C_0 - x_1 - x_2,$$

where $C_0$ is chosen such that

$$\exists \beta_0 > 0, \forall x \in \Omega, \beta(x) \geq \beta_0.$$

We also write

$$\beta_1 = \max_{x \in \Omega} \beta(x).$$
Lemma 6 The solution $y$ of (14), (15), (16) satisfies the following differential inequality

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \beta |y|^2 dx + \int_{\Omega} \beta |\nabla y|^2 dx + \frac{2}{\beta_1} \int_{\Omega} (N.U) \beta |y|^2 dx \leq \frac{4}{3} \int_{\Omega} |u_1|^3 dx. \quad (18)$$

Proof.

Multiply equation (14) by $\beta y$. We obtain, as $\Delta \beta = 0$ and $\frac{\partial \beta}{\partial x_1} + \frac{\partial \beta}{\partial x_2} = -2$,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \beta |y|^2 dx + \int_{\Omega} \beta |\nabla y|^2 dx + 2 \int_{\Omega} (N.U) |y|^2 dx + \frac{4}{3} \int_{\Omega} |y|^2 y dx = 0.$$  

Using Lemma 5 with $p = 3$ we obtain the desired result.
Notice that up to this point the control $\nu$ has not been chosen.

In the case when $\Gamma_0$ is empty which means that we can apply a control on the whole boundary, we don't have to take the boundary condition $w(t,0) = 0$ and we can take $w$ such that $\min_{x \in \Omega} U(t,x) \geq \min_{z \in (0,A)} w(t,z) \geq \alpha(t) > 0$ if $t > T_0$, which ensures that $U$ has a strictly positive minimum when $t > T_0$.

When $\Gamma_0$ is not empty, due to the boundary condition $w(t,0) = 0$ we cannot have a strictly positive minimum for $U$ over $\Omega$. 
Let us now make a choice for $w$ and $v$. On the interval $(T_0, T_0 + \delta)$, where $0 < \delta \leq \delta_0$, we set

$$w(t, z) = \frac{1}{\sqrt{(t - T_0)}}\left(e^{-\frac{(z-5A)^2}{8(t-T_0)}} - e^{-\frac{(z+5A)^2}{8(t-T_0)}}\right).$$

(19)

We can see that $w$ satisfies (9), (10) with a suitable control $v$ and (11).

For $0 < a \leq z \leq A$ we have

$$w(t, z) \geq w(t, a) = \frac{2}{\sqrt{(t - T_0)}}e^{-\frac{(a^2+25A^2)}{8(t-T_0)}}\sinh\left(\frac{5Aa}{4(t-T_0)}\right)$$

\[\geq \frac{5Aa}{2(t-T_0)^{\frac{3}{2}}}e^{-\frac{(a^2+25A^2)}{8(t-T_0)}}.\]
At the same time we also have
\[ \exists C_0 > 0, \forall a \in (0, A), \forall t \in (T_0, T_0 + \delta), \forall z, 0 \leq z \leq a, w(t, z) \leq w(t, a) \leq C_0 a. \]

We will write
\[ \Omega_a = \{ x \in \Omega, \ 0 \leq x_2 - x_1 \leq a \} \]
and we have
\[ |\Omega_a| \leq C a, \]
and
\[ \min_{x \in \Omega \setminus \Omega_a} U(t, x) \geq w(t, a) \geq \frac{5 A a}{2(t - T_0)^{3/2}} e^{-\frac{(a^2 + 25 A^2)}{8(t - T_0)}}. \]
Therefore, from (18), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \beta|y|^2 \, dx + \int_\Omega \beta|\nabla y|^2 \, dx + \frac{5NAa}{\beta_1(t-T_0)^{\frac{3}{2}}} e^{-\frac{(a^2+25A^2)}{8(t-T_0)}} \int_\Omega \beta|y|^2 \, dx \\
\leq \frac{4}{3} \int_\Omega |u_1|^3 \, dx + 2N \int_{\Omega_a} w(t,a)|y|^2 \, dx \\
\leq \frac{4}{3} \int_\Omega |u_1|^3 \, dx + 2Nw(t,a)|\Omega_a|^{\frac{1}{3}} \left( \int_\Omega |y|^3 \, dx \right)^{\frac{2}{3}} \\
\leq \frac{4}{3} \int_\Omega |u_1|^3 \, dx + CNa^{\frac{4}{3}} \left( \int_\Omega |u_1|^3 \, dx \right)^{\frac{2}{3}}.
\]
We now take
\[ a = \frac{1}{N^{\frac{3}{4}}} \]
which implies the following differential inequality
\[
\frac{d}{dt} \int_{\Omega} \beta |y|^2 dx \leq -\frac{10N^{\frac{1}{4}}A}{\beta_1(t-T_0)^{\frac{3}{2}}} e^{-\frac{26A^2}{8(t-T_0)/2}} \int_{\Omega} \beta |y|^2 dx + C(\|u_1\|_{L^3(\Omega)}).
\]
Using Gronwall Lemma, integrating this inequality on \((T_0, T_0 + \delta)\), we obtain
\[
\int_{\Omega} \beta |y(T_0 + \delta, x)|^2 dx \leq \left( \int_{\Omega} \beta |u_1|^2 dx \right) e^{-N^{\frac{1}{4}}g(\delta)} + \delta C(\|u_1\|_{L^3(\Omega)})
\]
where for \(\delta\) small enough
\[
g(\delta) = \int_{T_0}^{T_0+\delta} \frac{10A}{\beta_1(t-T_0)^{\frac{3}{2}}} e^{-\frac{26A^2}{8(t-T_0)/2}} dt \geq C e^{-\frac{A^2}{\delta}} > 0
\]
This implies
\[ \int_{\Omega} |y(T_0 + \delta, x)|^2 dx \leq \frac{\beta_1}{\beta_0} \|u_1\|_{L^2(\Omega)}^2 e^{-N\frac{1}{4} g(\delta)} + \frac{\delta}{\beta_0} C(\|u_1\|_{L^3(\Omega)}) \]
and, choosing first $\delta$ sufficiently small then $N$ sufficiently large we have proved the following

**Proposition 7** Given $u_1$ in $H^1_0(\Omega)$ (in fact $u_1 \in L^3(\Omega)$ would be enough), for every $\delta_0 > 0$ and for every $\epsilon_0 > 0$, there exists $\delta$ with $0 < \delta \leq \delta_0$ and there exists $N$ sufficiently large such that
\[ \|y(T_0 + \delta, \cdot)\|_{L^2(\Omega)} \leq \epsilon_0. \]

Now we choose the control $v$ on the time interval $(T_0 + \delta, T_0 + 2\delta_0)$ in (10) such that $w$ satisfies
\[ w(T_0 + 2\delta_0, \cdot) = 0. \]
This is possible using classical results on null controllability for the heat equation. Then we also have

\[ U(T_0 + 2\delta_0, \cdot) = 0. \]

Therefore,

\[ \|u(T_0 + 2\delta_0, \cdot)\|_{L^2(\Omega)} = \|y(T_0 + 2\delta_0, \cdot)\|_{L^2(\Omega)} \leq \|y(T_0 + \delta, \cdot)\|_{L^2(\Omega)} \leq \epsilon_0. \]

Notice that \( \epsilon_0 \) can be chosen as small as we wish. At this point we only know that the \( L^2(\Omega) \)-norm of \( u(T_0 + 2\delta_0, \cdot) \) is as small as we wish.
On the interval \((T_0 + 2\delta_0, T_0 + 3\delta_0)\) we let the system evolve freely and we take the boundary control equal zero. Then using the regularizing effect of Burgers equation we see that at time \(T_0 + 3\delta_0\) we have

\[ ||u(T_0 + 3\delta_0, \cdot)||_{H^1_0(\Omega)} \leq \epsilon_1, \]

where \(\epsilon_1\) can be taken as small as we wish provided \(\epsilon_0\) is small enough.

Therefore, on the time interval \((T_0 + 3\delta_0, T)\) we can use a result of local exact controllability to trajectories for 2-d Burgers equations (not completely trivial !) to find a boundary control \(h\) such that

\[ u(T, \cdot) = 0. \]
A situation without global controllability
Theorem 2 was proved under the restrictive assumption (7) on the boundary $\Gamma_0$. The next result shows that without this assumption the global controllability property may fail.

Let us suppose that the geometrical situation is such that there exists a function $\rho(x) \in C^2(\bar{\Omega})$ such that

$$\rho|_{\Gamma_1} = 0, \quad \rho(x) > 0 \text{ in } \Omega, \quad \frac{\partial \rho}{\partial x_1} + \frac{\partial \rho}{\partial x_2} < 0 \quad \forall x \in \bar{\Omega}. \quad (20)$$

Of course this cannot occur in the situation considered in the previous section, but there are many cases where such a function $\rho$ exists, for example when $\Omega = \{(x_1, x_2), \ 0 < x_2 - x_1 < 1, \ -1 < x_1 + x_2 < 1\}$ and $\Gamma_1 = \{(x_1, x_2), \ 0 < x_2 - x_1 < 1, \ x_1 + x_2 = 1\}$.

For a function $v$ defined on $\Omega$ or $(0, T) \times \Omega$ we set

$$v^+ = \max(v, 0), \ v^- = (-v)^+. \quad (33)$$
Theorem 8 Suppose that condition (20) holds true. Let $f \in L^2(0, T; L^2(\Omega))$ and $u_0 \in H^1_0(\Omega)$ such that $u_0 \neq 0$. Then there exists a time $T_0(u_0, f) > 0$ such that for each $T \leq T_0(u_0, f)$ there is no solution to problem (3)-(5) in the space $u \in H^{1,2}(Q)$ satisfying (6).

Proof. We argue by contradiction. Let $u_0 \in H^1_0(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$ be given functions. Suppose that there exists a solution $u$ to (3)-(6). Then we consider the function $y(t, x) = u(t, x) - u_0(x)$ which satisfies the following system of equations
\[
\frac{\partial y}{\partial t} - \Delta y + \frac{\partial y^2}{\partial x_1} + \frac{\partial y^2}{\partial x_2} + 2 \frac{\partial (yu_0)}{\partial x_1} + 2 \frac{\partial (yu_0)}{\partial x_2} = q \quad \text{in} \ (0, T) \times \Omega,
\]
\[y|_{\Gamma_0} = 0, \quad y|_{\Gamma_1} = h \quad y(0, \cdot) = 0, \]
\[y(T, \cdot) = -u_0, \]

where
\[q = \Delta u_0 - \frac{\partial u_0^2}{\partial x_1} - \frac{\partial u_0^2}{\partial x_2} + f.\]

We set
\[\rho_1(x) = \rho(x)^4.\]

Multiplying the equation by \(\rho_1 y^+\) and integrating by parts we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_1 |y^+|^2 \, dx + \int_{\Omega} \left( \rho_1 |\nabla y^+|^2 - \frac{\Delta \rho_1}{2} |y^+|^2 - \frac{2}{3} \left( \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2} \right)(y^+)^3 \right) \, dx \\
+ \int_{\Gamma_0} \frac{1}{2} \frac{\partial \rho_1}{\partial n} |y^+|^2 \, d\sigma - 2 \int_{\Omega} \left( \left( \frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2} \right) \rho_1 u_0 y^+ - u_0 \left( \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2} \right) |y^+|^2 \right) \, dx \\
= \int_{\Omega} f \rho_1 y^+ \, dx - \int_{\Omega} \nabla u_0 \cdot \nabla y^+ \rho_1 \, dx - \int_{\Omega} \nabla u_0 \cdot \nabla \rho_1 y^+ \, dx \\
+ \int_{\Omega} u_0^2 y^+ \left( \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2} \right) \, dx + \int_{\Omega} u_0^2 \rho_1 \left( \frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2} \right) \, dx \\
\leq \int_{\Omega} f \rho_1 y^+ \, dx - \int_{\Omega} \nabla u_0 \cdot \nabla y^+ \rho_1 \, dx - \int_{\Omega} \nabla u_0 \cdot \nabla \rho_1 y^+ \, dx \\
+ \int_{\Omega} u_0^2 \rho_1 \left( \frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2} \right) \, dx.
\]
By (20) we have $\int_{\Gamma_0} \frac{1}{2} \frac{\partial \rho_1}{\partial n} |y^+|^2 d\sigma = 0$. Again using (20) we may assume that for some positive constant $M$ we have $-\frac{2}{3} \left( \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2} \right) > M \rho_1^\frac{3}{4}$ for all $x \in \bar{\Omega}$. Then denoting by $C_i$ various constants independent of $y$ and $u_0$ we have

$$\int_\Omega \left( -\frac{\Delta \rho_1}{2} |y^+|^2 - \frac{2}{3} \left( \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2} \right) (y^+)^3 \right) dx \geq \int_\Omega \left( -C_0 \rho_1^\frac{1}{2} |y^+|^2 + M \rho_1^\frac{3}{4} (y^+)^3 \right) dx$$

$$\geq -C_1 \left( \int_\Omega \rho_1^\frac{3}{4} (y^+)^3 dx \right)^\frac{2}{3} + M \int_\Omega \rho_1^\frac{3}{4} (y^+)^3 dx \geq \frac{3M}{4} \int_\Omega \rho_1^\frac{3}{4} (y^+)^3 dx - C_2.$$
Then we have

\[ 2 \int_{\Omega} \left( \frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2} \right) \rho_1 u_0 y^+ \, dx \leq \frac{1}{4} \int_{\Omega} \rho_1 |\nabla y^+|^2 \, dx + C_3 \int_{\Omega} u_0^2 \rho_1 |y^+|^2 \, dx \]

\[ \leq \frac{1}{4} \int_{\Omega} \rho_1 |\nabla y^+|^2 \, dx + C_4 \|u_0\|_{H^1_0(\Omega)}^2 \left( \int_{\Omega} \rho_1^\frac{3}{4} (y^+) \, dx \right)^\frac{2}{3} \]

\[ \leq \frac{1}{4} \int_{\Omega} \rho_1 |\nabla y^+|^2 \, dx + \frac{M}{4} \int_{\Omega} \rho_1^\frac{3}{4} (y^+) \, dx + C_5 \|u_0\|^6_{H^1_0(\Omega)}. \]

Also

\[ 2 \int_{\Omega} u_0 \left( \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2} \right) |y^+|^2 \, dx \leq C_6 \|u_0\|_{H^1_0(\Omega)} \left( \int_{\Omega} \rho_1^\frac{3}{4} (y^+) \, dx \right)^\frac{2}{3} \]

\[ \leq \frac{M}{4} \int_{\Omega} \rho_1^\frac{3}{4} (y^+) \, dx + C_7 \|u_0\|^3_{H^1_0(\Omega)}. \]
We also obtain

\[
\int_\Omega f \rho_1 y^+ dx - \int_\Omega \nabla u_0 \cdot \nabla y^+ \rho_1 dx \\
- \int_\Omega \nabla u_0 \cdot \nabla \rho_1 y^+ dx + \int_\Omega u_0^2 \rho_1 (\frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2}) dx \\
\leq C_8(\|f\|_{L^2(Q)}^2 + \|u_0\|_{H^1_0(\Omega)}^2 + \|u_0\|_{H^1_0(\Omega)}^4) \\
+ \frac{1}{2} \int_\Omega \rho_1 |y^+|^2 dx + \frac{1}{4} \int_\Omega \rho_1 |\nabla y^+|^2 dx.
\]

Using all these inequalities we obtain

\[
\frac{d}{dt} \int_\Omega \rho_1 |y^+|^2 dx + \int_\Omega \rho_1 |\nabla y^+|^2 dx + \int_\Omega \frac{M}{2} \rho_1^\frac{3}{4} (y^+)^3 dx \\
\leq C_9(1 + \|f\|_{L^2(Q)}^2 + \|u_0\|_{H^1_0(\Omega)}^2 + \|u_0\|_{H^1_0(\Omega)}^6) + \int_\Omega \rho_1 |y^+|^2 dx.
\]
Applying Gronwall’s inequality we obtain, as $y^+(0, \cdot) = 0$,

$$
\sup_{t \in (0, T)} \int_{\Omega} \rho_1 |y^+|^2 \, dx \leq C_{10} (1 + \|f\|_{L^2(Q)}^2 + \|u_0\|_{H^1_0(\Omega)}^2 + \|u_0\|_{H^1_0(\Omega)}^6) T e^T.
$$

Since the right hand side goes to zero as $T$ goes to zero and $y^+(T) = u_0^-$, we immediately arrive to a contradiction and the proof of Theorem 8 is complete.