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THE DISCONTINUITY SET OF SOLUTIONS OF THE TV DENOISING PROBLEM AND SOME EXTENSIONS*

VICENT CASELLES†, ANTONIN CHAMBOLLE‡, AND MATTEO NOVAGA§

Abstract. The main purpose of this paper is to prove that the jump discontinuity set of the solution of the total variation based denoising problem is contained in the jump set of the datum to be denoised. We also prove some extensions of this result for the total variation minimization flow, for anisotropic norms, and for some more general convex functionals, which include the minimal surface equation case and its anisotropic extensions.

Key words. total variation, jump discontinuity set, regularity

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1. Introduction.

The use of total variation as a regularization term for image denoising and restoration was introduced by Rudin, Osher, and Fatemi in [30]. If Ω denotes the image domain, when dealing with the restoration problem one minimizes the total variation functional

$$u \;\mapsto\; \int_\Omega |Du| \tag{1.1}$$

under some constraints which model the process of image acquisition, including blur and noise. The constraint can be written as $f = K * u + n$, where $f \in L^2(\Omega)$ is the observed image, K is a convolution operator whose kernel represents the point spread function of the optical system, n is the noise (typically a white Gaussian noise of zero mean), and u is the ideal image, previous to distortion. The denoising problem corresponds to $K = I$, and, in this case, the constraint becomes

$$f \;=\; u + n. \tag{1.2}$$

In practice, the only information we have about the noise is statistical. Assuming that n is a Gaussian white noise of zero mean and standard deviation σ, the constraint (1.2) can be imposed in an integral form as

$$\int_\Omega (f-u)^2\,dx \;\le\; \sigma^2|\Omega|, \tag{1.3}$$

where σ^2 denotes a bound on the noise variance. Among all images satisfying this constraint, the denoised image is chosen as the one minimizing (1.1) [30]. As proved

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†Departament de Tecnologia, Universitat Pompeu Fabra, Passeig de Circumval.lació 8, 08003 Barcelona, Spain (vicent.caselles@upf.edu). This author's research was partially supported by the PNPGC project, reference MTM2006-14836.

‡CMAP, École Polytechnique, CNRS, 91128 Palaiseau, France (antonin.chambolle@polytechnique.fr). This author's research was partially supported by the ANR project "MICA" (Mouvements d'Interfaces, Calculs et Applications).

§Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy (novaga@dm.unipi.it).

by Chambolle and Lions in [20], minimizing (1.1) under the constraint (1.3) amounts to solving

$$\min_{u\in BV(\Omega)}\left\{\int_\Omega |Du| + \frac{1}{2\lambda}\int_\Omega (u-f)^2\,dx\right\} \tag{1.4}$$

for some Lagrange multiplier $\lambda > 0$. When the noise bound is not known, λ acts as a penalization term.

One of the main features of total variation denoising (1.4), confirmed by numerical experiments, is its ability to restore the discontinuities of the image [30, 20, 23]. The a priori assumption is that functions of bounded variation (the BV model [8]) are a reasonable functional model for many problems in image processing, in particular, for denoising and restoration problems. Typically, functions of bounded variation admit a set of discontinuities which is countably rectifiable [8], being continuous in some sense (in the measure theoretic sense) away from discontinuities. The discontinuities could be identified with edges. The ability of total variation regularization to recover edges is one of the main features which advocates for the use of this model which had a strong influence in the use of BV functions in image processing (its ability to describe textures is less clear, even if some textures can be recovered, up to a certain scale of oscillation).

The main purpose of this paper is to prove that the jump discontinuities of the solution u of the denoising problem (1.4) are contained in the jump discontinuities of the datum f, assuming that $f \in BV(\Omega)\cap L^\infty(\Omega)$. Partial information on this question was known through the computation of explicit solutions in several works [31, 13, 28, 14, 5, 6]. In particular, let us mention the full description of the solution in the case that $f = \chi_C$, where C is a convex subset of $\mathbb{R}^N$, $N \geq 2$ [5, 6, 4]. In this case, it is clear from the explicit solution that the jump set of the solution u is contained in ∂C, and it coincides with it when ∂C is of class $C^{1,1}$ and λ is small enough. When $N = 2$, a more detailed analysis, given in [5], also proves that the solution is $W^{1,1}$ inside C, being 0 outside. Other explicit solutions for piecewise constant data f made of sums of characteristic functions of convex sets were given in [14]. The case of solutions when f has a radial symmetry can be found in [9, 11, 28, 31]. The picture coming out from these works is completed with the main result of this paper.

Let us mention that our result gives some information about the nature of the "staircasing effect." Staircasing, i.e., the creation of image flat regions separated by boundaries, is one of the observed artifacts which appear in total variation image denoising. The most obvious example is when denoising a smooth ramp plus noise (see Figure 1.1). In the discrete framework, this effect has been reported to be a consequence of the nondifferentiability of the total variation norm when the gradient vanishes [29]. Indeed, this reason is at the origin of the appearance of flat regions at points where the gradient vanishes, as is shown by explicit solutions in the radially symmetric case [9, 11, 28, 31] as well as in one dimension (see below and Figure 1.1). We also believe that this is the correct explanation in the continuous framework (see, for instance, [5]).

But our result says that, at the continuous level, no new jump discontinuities may appear in the solution that were not present in the (BV) datum f. Hence, if the original signal f is smooth enough, one expects that flat areas will appear, but they should not be, strictly speaking, separated by jumps (however, steep transitions between flat areas might look close to being jumps and still look like a "staircase"). Observe, for instance, that if $\Omega = (0,1)$, $f : (0,1) \to \mathbb{R}$ is a smooth oscillating ramp

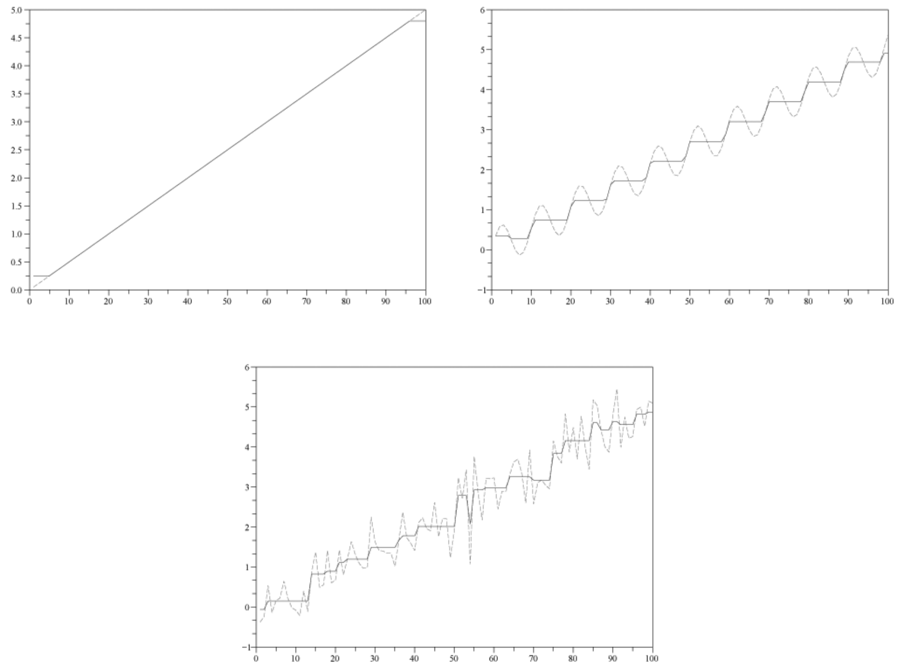

FIG. 1.1. *Top, left: a monotonous ramp and its BV-regularization; observe that nothing happens except at the boundaries. Right: staircasing in a (smooth) sinusoidal ramp (see explicit computation in the text). Bottom: a noisy ramp; the staircasing effect is maximal.*

(for instance, the function $x + .1\sin(100x)$), then it is easy to show that, actually, u does not present any discontinuity and the "staircasing" effect is reduced to a flattening of u near the local extrema of f. Indeed, the Euler–Lagrange equation for (1.4) turns out to be

$$-\lambda\phi'(x) \;+\; u(x) \;=\; f(x),$$

where $-1 \le \phi \le 1$ a.e., and $\phi u' = |u'|$. First, by the maximum principle and standard techniques, one easily shows that u is Lipschitz, with $\|u'\|_\infty \le \max_{[0,1]} |f'|$ (from which we deduce that ϕ is $C^{1,1}$). Then one easily deduces that u must be constant on each interval of $\{u \neq f\}$ (since there ϕ' does not vanish, and hence $\phi \notin \{-1, 1\}$). Hence, when u is not constant, $u = f$ (which is easily seen from the equation, since then ϕ must be identically 1, if u increases, or -1 if u decreases), and if $u \neq f$, u is flat.

On the other hand, if we are given some discrete noisy data, we could interpret it both as a BV data with high norm and discontinuity around each pixel (although this point of view is a bit strange) or as a non-BV data: in both cases, a strong staircasing effect is compatible with our result (and we find that the total variation flow will reduce progressively the number of discontinuities, in particular, by a progressive merging of the flat areas). See Figures 1.1 and 1.2 for numerical experiments illustrating these comments.

The main result of the paper is extended in several directions. First, we prove a similar statement for the solutions of the gradient descent flow of the total variation, starting from $f \in L^\infty(\Omega)$. In this case, using nonlinear semigroup theory, we have a partial answer: the jump discontinuity set $J_{u(t)}$ of the solution $u(t)$ is contained in the jump set J_f of f when f is BV and lies in the closure of the domain of the

FIG. 1.2. *Top: an image, original and noisy. Bottom: the corresponding total variation-regularized image (same λ). Observe that the staircasing is hardly visible on the regularized version of the original image, which is "closer" to BV than the noisy version.*

operator $-\operatorname{div}\left(\frac{Du}{|Du|}\right)$ in $L^\infty(\Omega)$. If f is just bounded, we get only that $J_{u(t)} \subseteq J_{u(s)}$ for any $t \geq s > 0$. Other extensions concern the case of several boundary conditions or anisotropic total variation norms. Eventually, we also see how the above results can be extended to convex functionals with linear growth, of the form $F(\xi) = \phi(\xi, -1)$, $\xi \in \mathbb{R}^N$, where $\phi : \mathbb{R}^{N+1} \to \mathbb{R}$ is a smooth and elliptic norm on $\mathbb{R}^{N+1}$. This includes, in particular, the case where $F(\xi) = \sqrt{1 + |\xi|^2}$, $\xi \in \mathbb{R}^N$, which is more carefully analyzed.

Let us describe the plan of the paper. In section 2 we recall some basic facts about functions of bounded variation. In section 3 we prove the main result of the paper concerning the jumps of the solutions of the denoising problem (1.4). We then extend this result to the case of the total variation flow (section 4). We discuss in section 5 the extension of our results to similar problems (other boundary conditions, anisotropic norms, or more general convex functionals as described in our last paragraph).

2. Notation and preliminaries on *BV* functions. Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector-valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\sup\left\{\int_\Omega u \operatorname{div} z \, dx : z \in C_0^\infty(\Omega; \mathbb{R}^N),\ |z(x)| \leq 1\ \forall x \in \Omega\right\}, \tag{2.1}$$

where for a vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$, and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. The map $u \to |Du|(\Omega)$ is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous. $BV(\Omega)$ is a Banach space when endowed with the norm $\|u\| := \int_\Omega |u|\,dx + |Du|(\Omega)$.

A measurable set $E \subseteq \Omega$ is said to be of finite perimeter in Ω if (2.1) is finite when u is substituted with the characteristic function χ_E of E. The perimeter of E in Ω is defined as $P(E,\Omega) := |D\chi_E|(\Omega)$. We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, respectively, the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

Let $u \in [L^1_{\mathrm{loc}}(\Omega)]^m$ ($m \geq 1$). We say that u has an approximate limit at $x \in \Omega$ if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |u(y) - z|\,dy = 0. \tag{2.2}$$

The set of points where this does not hold is called the approximate discontinuity set of u, and it is denoted by S_u. Using Lebesgue's differentiation theorem, one can show that the approximate limit z exists at $\mathcal{L}^N$-a.e. $x \in \Omega$ and is equal to $u(x)$: in particular, $|S_u| = 0$.

If $x \in \Omega \setminus S_u$, the vector z is uniquely determined by (2.2), and we denote it by $\tilde{u}(x)$. We say that u is approximately continuous at x if $x \notin S_u$ and $\tilde{u}(x) = u(x)$, that is, if x is a Lebesgue point of u (with respect to the Lebesgue measure). Let $u \in [L^1_{\mathrm{loc}}(\Omega)]^m$ and $x \in \Omega \setminus S_u$; we say that u is approximately differentiable at x if there exists an $m \times N$ matrix L such that

$$\lim_{\rho \downarrow 0} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{\rho}\,dy \;=\; 0. \tag{2.3}$$

In that case, the matrix L is uniquely determined by (2.3) and is called the approximate differential of u at x.

For $u \in BV(\Omega)$, the gradient Du is an N-dimensional Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u\,dx$, where ∇u is the Radon–Nikodym derivative of the measure Du with respect to the Lebesgue measure in $\mathbb{R}^N$. The function u is approximately differentiable $\mathcal{L}^N$-a.e. in Ω, and the approximate differential coincides with $\nabla u(x)$ $\mathcal{L}^N$-a.e. The singular part $D^s u$ can be also split into two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. We say that $x \in \Omega$ is an *approximate jump point of* u if there exist $u^+(x) \neq u^-(x) \in \mathbb{R}$ and $|\nu_u(x)| = 1$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{|B^+_\rho(x,\nu_u(x))|} \int_{B^+_\rho(x,\nu_u(x))} |u(y) - u^+(x)|\,dy = 0,$$

$$\lim_{\rho \downarrow 0} \frac{1}{|B^-_\rho(x,\nu_u(x))|} \int_{B^-_\rho(x,\nu_u(x))} |u(y) - u^-(x)|\,dy = 0,$$

where $B^+_\rho(x,\nu_u(x)) = \{y \in B(x,\rho) : \langle y - x, \nu_u(x)\rangle > 0\}$ and $B^-_\rho(x,\nu_u(x)) = \{y \in B(x,\rho) : \langle y - x, \nu_u(x)\rangle < 0\}$. We denote by J_u the set of approximate jump points of u. If $u \in BV(\Omega)$, the set S_u is countably $\mathcal{H}^{N-1}$ rectifiable, J_u is a Borel subset of S_u, and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ [8]. In particular, we have that $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ is either a point of approximate continuity of $\tilde{u}$ or a jump point with two limits in the above sense. Eventually, we have

$$D^j u = D^s u \llcorner_{J_u} = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner_{J_u} \quad \text{and} \quad D^c u = D^s u \llcorner_{(\Omega \setminus S_u)}.$$

If $E \subseteq \mathbb{R}^N$ is a measurable set and $x \in \mathbb{R}^N$, we define the upper (resp., lower) density of E at x by

$$\overline{D}(E,x) := \limsup_{r\to 0} \frac{|E\cap B(x,r)|}{|B(x,r)|} \qquad \left(\text{resp.,} \quad \underline{D}(E,x) := \liminf_{r\to 0} \frac{|E\cap B(x,r)|}{|B(x,r)|}\right).$$

If $u \in BV(\Omega)$ and we define

$$u^*(x) := \inf\{t : \overline{D}(\{u>t\},x)\} \quad \text{and} \quad u_*(x) := \sup\{t : \underline{D}(\{u<t\},x)\},$$

then u is approximately continuous at $x \in \Omega$ if and only if $u^*(x) = u_*(x)$. Moreover, $u^*(x) = u^+(x)$ and $u_*(x) = u^-(x)$ $\mathcal{H}^{N-1}$ a.e. on J_u.

For a comprehensive treatment of functions of bounded variation, we refer the reader to [8].

3. The discontinuities of solutions of the total variation denoising problem. Given a function $f \in L^2(\Omega)$ and $\lambda > 0$ we consider the minimum problem

$$\begin{aligned} &\min_{u\in BV(\Omega)} \mathcal{F}_\lambda(u), \quad \text{with} \\ &\mathcal{F}_\lambda(u) \;=\; \int_\Omega |Du| + \frac{1}{2\lambda}\int_\Omega (u-f)^2\,dx. \end{aligned} \tag{3.1}$$

Notice that problem (3.1) always admits a unique solution u_λ, since the functional $\mathcal{F}_\lambda$ is strictly convex.

Let us recall the following observation, which is proved in [21, 6] (see also [19, 15]).

PROPOSITION 3.1. *For any $t \in \mathbb{R}$, consider the minimal surface problem*

$$\min_{E\subseteq\Omega} P(E,\Omega) + \frac{1}{\lambda}\int_E (t - f(x))\,dx \tag{3.2}$$

(whose solution is defined in the class of finite-perimeter sets and hence up to a Lebesgue-negligible set). Then, $\{u_\lambda > t\}$ (resp., $\{u_\lambda \geq t\}$) is the minimal (resp., maximal) solution of (3.2). *In particular, for all t but a countable set, the solution of this problem is unique.*

A proof that $\{u_\lambda > t\}$ and $\{u_\lambda \geq t\}$ both solve (3.2) is found in [21, Prop. 2.2]. A complete proof of this proposition, which we do not give here, follows from the coarea formula, which shows that

$$\mathcal{F}_\lambda(u) \sim \int \left(P\left(\{u>t\},\Omega\right) + \frac{1}{\lambda}\int_{\{u>t\}} (t-f)\,dx \right) dt,$$

and from the following comparison result for solutions of (3.2) which is proved in [6, Lemma 4].

LEMMA 3.2. *Let $f, g \in L^1(\Omega)$ and E and F be, respectively, minimizers of*

$$\min_E P(E,\Omega) - \int_E f(x)\,dx \quad \text{and} \quad \min_F P(F,\Omega) - \int_F g(x)\,dx.$$

Then, if $f < g$ a.e., $|E\setminus F| = 0$ (in other words, $E \subseteq F$ up to a negligible set).

The proof of this last lemma relies only on the inequality $P(A\cup B,\Omega) + P(A\cap B,\Omega) \leq P(A,\Omega) + P(B,\Omega)$ and is easily generalized to other situations (Dirichlet boundary conditions, anisotropic and/or nonlocal perimeters, etc.; see the proof in [6]—see also [1] for a similar general statement).

Eventually, we mention that the result of Proposition 3.1 remains true if the term $(u(x)-f(x))^2/(2\lambda)$ in (3.1) is replaced with a term of the form $\Psi(x,u(x))$, with Ψ of class C^1 and strictly convex in the second variable, and replacing $(t-f(x))/\lambda$ with $\partial_u\Psi(x,t)$ in (3.2). A variant of this principle for $\Psi = |\cdot|$ is found in [22]

From Proposition 3.1 and the regularity theory for minimal surfaces (see, for instance, [7, 27]), we obtain the following regularity result (see also [1]).

COROLLARY 3.3. *Let $f \in L^p(\Omega)$, with $p > N$. Then, for all $t \in \mathbb{R}$, the superlevel set $E_t := \{u_\lambda > t\}$ (resp., $\{u_\lambda \geq t\}$) has boundary of class $C^{1,\alpha}$, for all $\alpha < (p-N)/p$, out of a closed singular set Σ of Hausdorff dimension at most $N-8$. Moreover, if $p = \infty$, the boundary of E_t is of class $W^{2,q}$ out of Σ, for all $q < \infty$, and is of class $C^{1,1}$ if $N = 2$.*

We now show that the jump set of u_λ is always contained in the jump set of f.

THEOREM 3.4. *Let $f \in BV(\Omega) \cap L^\infty(\Omega)$. Then, for all $\lambda > 0$,*

$$J_{u_\lambda} \subseteq J_f \tag{3.3}$$

(up to a set of zero $\mathcal{H}^{N-1}$-measure).

Proof. Let $E_t := \{u_\lambda > t\}$, and let Σ_t be its singular set given by Corollary 3.3. We show that for all $t_1 \neq t_2$ there holds

$$\mathcal{H}^{N-1}\left(\partial E_{t_1} \cap \partial E_{t_2} \setminus J_f\right) = 0. \tag{3.4}$$

Suppose by contradiction that (3.4) does not hold for some values $t_1 < t_2$, and let $x \in \partial E_{t_1} \cap \partial E_{t_2} \setminus J_f$. We can assume that x does not belong to $\Sigma_{t_1} \cup \Sigma_{t_2}$. Therefore, by Corollary 3.3, we know that both ∂E_{t_1} and ∂E_{t_2} are regular in a neighborhood of x; therefore we may write the set ∂E_{t_i} locally as the graph of a function $v_i \in W^{2,p}(U)$, $i \in \{1,2\}$, where U is a neighborhood of x in the tangent space to ∂E_{t_i} at x (which we identify with $\mathbb{R}^{N-1}$). In this way, the Euler–Lagrange equation for (3.2) becomes

$$\operatorname{div} \frac{\nabla v_i(y)}{\sqrt{1+|\nabla v_i(y)|^2}} + \frac{1}{\lambda}\left(t_i - f(y, v_i(y))\right) = 0, \qquad y \in U. \tag{3.5}$$

From $t_1 < t_2$ and Lemma 3.2, it follows that $E_{t_2} \subseteq E_{t_1}$, which in turn gives $v_2 \geq v_1$ in U. Recall that since $f \in BV(\Omega)$, $\mathcal{H}^{N-1}$-a.e. $x \notin J_f$ is a Lebesgue point for f [8]. Hence, without loss of generality, we may also assume that x is a Lebesgue point for f and, also, a point of approximate differentiability for both v_i and ∇v_i, $i \in \{1,2\}$. In particular, (3.5) has a pointwise meaning at x, and there holds $v_1(x) = v_2(x) = 0$ and $\nabla v_1(x) = \nabla v_2(x) = 0$. As a consequence, subtracting the two equations satisfied by v_1 and v_2 at x, we obtain

$$\Delta v_1(x) - \Delta v_2(x) = \frac{t_2 - t_1}{\lambda} > 0,$$

which contradicts the inequality $v_2 \geq v_1$. ◻

Remark 1. Notice that if f is continuous at $x \in \partial E_{t_1} \cap \partial E_{t_2}$, reasoning as in the proof of Theorem 3.4 it follows that $x \in \Sigma_{t_1} \cup \Sigma_{t_2}$. Indeed, using the continuity of f we can choose the neighborhood U small enough such that there exist two constants c_1, c_2 with the property

$$\operatorname{div} \frac{\nabla v_1(y)}{\sqrt{1+|\nabla v_1(y)|^2}} \geq c_1 > c_2 \geq \operatorname{div} \frac{\nabla v_2(y)}{\sqrt{1+|\nabla v_2(y)|^2}}, \qquad y \in U, \tag{3.6}$$

which contradicts $v_2 \geq v_1$ as above.

In particular, if $N \le 7$ and $f \in C(B_\rho(x)) \subset \Omega$, then $u_\lambda \in C(B_\rho(x))$.

Remark 2. By a result of Calderón [18], if $p > N$, any function $v \in W^{1,p}(\Omega)$ is differentiable a.e. (this reduces to Rademacher's theorem when $p = +\infty$). We could have used this result in the proof of Theorem 3.4, but we used the simpler result that both v_i and ∇v_i are a.e. approximately differentiable. In any case we have that $D^2(v_2 - v_1)(x) \ge 0$, since $v_2 - v_1$ has a minimum at x and ∇v_1 and ∇v_2 are approximately differentiable at x.

4. The total variation flow. To fix ideas, let us assume in this section that Ω is an open bounded set with Lipschitz boundary. Let us consider the minimizing total variation flow

$$\begin{aligned} &\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{Du}{|Du|}\right) && \text{in } Q_T =]0,T[\times\Omega, \\ &\frac{Du}{|Du|}\cdot\nu^\Omega = 0 && \text{in } Q_T =]0,T[\times\partial\Omega \end{aligned} \tag{4.1}$$

with the initial condition

$$u(0,x) = f(x), \qquad x \in \Omega. \tag{4.2}$$

Let us recall that, in the Hilbertian framework (in L^2), it is the gradient flow of the total variation as defined in [17]. In the general case we shall follow [9, 13]. The purpose of this section is to prove the following result.

THEOREM 4.1. *Let $f \in L^N(\Omega)$. Let $u(t)$ be the solution of* (4.1) *with initial condition $u(0,x) = f(x)$. Then $u(t) \in L^\infty(\Omega) \cap BV(\Omega)$ for any $t > 0$, and*

$$J_{u(t)} \subseteq J_{u(s)} \qquad \forall t > s > 0. \tag{4.3}$$

Moreover, if $u(s)$ is continuous at $x \in \Omega$, then so is $u(t)$ for any $t > s > 0$. If $f \in \overline{\mathrm{Dom}(\mathcal{A}_\infty)} \cap BV(\Omega)$, then the above assertions are true up to $s = 0$.

The operator $\mathcal{A}_\infty$ is the intersection of the (L^2-) subdifferential of the total variation with L^∞ and is defined precisely later; see Definition 4.2.

To prove Theorem 4.1, let us recall some basic facts about the operator $-\operatorname{div}\left(\frac{Du}{|Du|}\right)$ in L^p spaces. Since it suffices for our purposes, we shall consider only the case $p \in [\frac{N}{N-1}, \infty]$. For any $p \in [1,\infty]$, let us define the space

$$X(\Omega)_p := \left\{ z \in L^\infty(\Omega, \mathbb{R}^N) \ : \ \operatorname{div}(z) \in L^p(\Omega) \right\}.$$

If $z \in X(\Omega)_p$ and $w \in BV(\Omega) \cap L^q(\Omega)$, $p^{-1} + q^{-1} = 1$, we define the functional $(z \cdot Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z\cdot Dw), \varphi\rangle := -\int_\Omega w\,\varphi \operatorname{div} z\,dx - \int_\Omega w\,z\cdot\nabla\varphi\,dx.$$

Then $(z \cdot Dw)$ is a Radon measure in Ω, and $(z\cdot Dw) = z\cdot\nabla w$ if $w \in W^{1,1}(\Omega)\cap L^q(\Omega)$.

Finally, we observe that (see [12]) if $z \in X(\Omega)_p$, then there exists a function $[z\cdot\nu^\Omega] \in L^\infty(\partial\Omega)$ satisfying $\|[z\cdot\nu^\Omega]\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$ and such that for any $u \in BV(\Omega)\cap L^q(\Omega)$ we have

$$\int_\Omega u \operatorname{div} z\;dx + \int_\Omega (z\cdot Du) = \int_{\partial\Omega} [z\cdot\nu^\Omega] u\,d\mathcal{H}^{N-1}.$$

DEFINITION 4.2. *We define the operator* $\mathcal{A}_p \subseteq L^p(\Omega) \times L^p(\Omega)$, $\frac{N}{N-1} \le p \le \infty$, *by*

$$(u,v) \in \mathcal{A}_p \quad \text{if and only if } u, v \in L^p(\Omega),\ u \in BV(\Omega), \text{ and}$$
$$\text{there exists } z \in X(\Omega)_p \text{ with } \|z\|_\infty \le 1 \text{ such that } (z \cdot Du) = |Du|,\ [z \cdot \nu^\Omega] = 0, \text{ and}$$
$$v = -\operatorname{div}(z) \qquad \text{in } \mathcal{D}'(\Omega).$$

By $v \in \mathcal{A}_p u$ we mean that $(u,v) \in \mathcal{A}_p$. By $L^1_w(]0,T[;BV(\Omega))$ we denote the space of weakly measurable functions $w : [0,T] \to BV(\Omega)$ (i.e., $t \in [0,T] \to \langle w(t), \phi\rangle$ is measurable for any $\phi \in BV(\Omega)^*$, where $BV(\Omega)^*$ denote the dual of $BV(\Omega)$) such that $\int_0^T \|w(t)\|\,dt < \infty$.

DEFINITION 4.3. *A function* $u \in C([0,T]; L^p(\Omega))$ *is called a strong solution of* (4.1) *if* $u \in W^{1,1}_{\rm loc}(0,T;L^p(\Omega)) \cap L^1_w(]0,T[;BV(\Omega))$ *and there exists* $z \in L^\infty\big(]0,T[\times\Omega;\mathbb{R}^N\big)$ *with* $\|z\|_\infty \le 1$ *such that*

$$\int_\Omega (z(t)\cdot Du(t)) = \int_\Omega |Du(t)| \qquad \text{for a.e. } t > 0, \tag{4.4}$$

$$[z(t)\cdot \nu^\Omega] = 0 \qquad \text{in } \partial\Omega \text{ for a.e. } t > 0, \tag{4.5}$$

and

$$u_t = \operatorname{div} z \qquad \text{in } \mathcal{D}'\left(]0,T[\times\Omega\right).$$

PROPOSITION 4.4. *The operator* $\mathcal{A}_p$ *is m-accretive in* $L^p(\Omega)$*; that is, for any* $f \in L^p(\Omega)$ *and any* $\lambda > 0$ *there is a unique solution* $u \in L^p(\Omega)$ *of the problem*

$$u + \lambda \mathcal{A}_p u \ni f. \tag{4.6}$$

Moreover, if $u_1, u_2 \in L^p(\Omega)$ *are the solutions of* (4.6) *corresponding to the right-hand sides* $f_1, f_2 \in L^p(\Omega)$, *then*

$$\|u_1 - u_2\|_p \le \|f_1 - f_2\|_p.$$

Moreover, the domain of $\mathcal{A}_p$ *is dense in* $L^p(\Omega)$ *when* $p < \infty$.

We denote by $\mathcal{R}_\lambda f$ the solution of (4.6), and by $\mathcal{R}^k_\lambda f$ its k-iterate, for any $k \ge 1$. Recall the notion of strong solution for nonlinear semigroups generated by accretive operators.

DEFINITION 4.5. *A function* u *is called a strong solution of in the sense of semigroups of* $\frac{du}{dt} + \mathcal{A}_p u \ni 0$ *with* $u(0) = f$ *if*

$$\begin{cases} u \in C([0,T];L^p(\Omega)) \cap W^{1,1}_{loc}(]0,T[;L^p(\Omega)), \\ u(t) \in \operatorname{Dom}(\mathcal{A}_p) \quad \text{a.e. in } t>0 \text{ and } u' + \mathcal{A}_p u(t) \ni 0 \text{ a.e. } t \in\,]0,T[, \\ u(0) = f. \end{cases} \tag{4.7}$$

Using Proposition 4.4, by Crandall and Ligget's semigroup generation theorem [24] we obtain the following result.

THEOREM 4.6. *Let* $f \in L^p(\Omega)$ *if* $\frac{N}{N-1} \le p < \infty$, *or let* $f \in \overline{\operatorname{Dom}(\mathcal{A}_\infty)}$ *if* $p = \infty$. *Then there is a unique strong solution in the sense of semigroups* $u(t) = S(t)f := \lim_{\lambda\downarrow 0,\, k\lambda \to t} \mathcal{R}^k_\lambda f \in C([0,T], L^p(\Omega))$ *of the problem*

$$\frac{du}{dt} + \mathcal{A}_p u \ni 0, \qquad u(0) = f. \tag{4.8}$$

Moreover, the semigroup solution is a strong solution of (4.1)*, and, conversely, any strong solution of* (4.1) *is a strong solution in the sense of semigroups of* (4.8).

Remark 3. Notice that given $p \in [\frac{N}{N-1}, \infty]$ the limit $\lim_{\lambda \downarrow 0,\, k\lambda \to t} \mathcal{R}_\lambda^k f$ is taken in $L^p(\Omega)$.

To prove Theorem 4.1, we need the following lemma.

Lemma 4.7. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions in $BV(\Omega) \cap L^\infty(\Omega)$. Assume that $J_{u_n} \subseteq J_{u_0}$, for all $n \in \mathbb{N}$, and $u_n \to u$ strongly in $L^\infty(\Omega)$. Then $\mathcal{H}^{N-1}$-almost every point of $\Omega \setminus J_{u_0}$ is a Lebesgue point for u. In particular, if $u \in BV(\Omega)$, then $J_u \subseteq J_{u_0}$. Moreover, if all the functions u_n are continuous at $x \in \Omega$, then also u is continuous at x.*

Proof. The thesis follows observing that if $x \in \Omega$ is a Lebesgue point for all the functions u_n, then it is also a Lebesgue point for u, and the same is true for a continuity point. □

Proof of Theorem 4.1.

Step 1. Assume that $f \in \overline{\mathrm{Dom}(\mathcal{A}_\infty)} \cap BV(\Omega)$. Then we know that $\mathcal{R}_\lambda^k f \to u(t)$ when $\lambda \to 0^+$ and $k\lambda \to t$ [24]. Then the result follows as a consequence of Theorem 3.4, Remark 3, and Lemma 4.7.

Step 2. Let $f \in L^\infty(\Omega)$. Observe that the functions $u(t) = S(t)f \in C([0,T]; L^\infty(\Omega))$ and $u(t) \in BV(\Omega)$ for any $t > 0$. Moreover, recall the following estimate, a consequence of the 0-homogeneity of the operator $\mathcal{A}_\infty$ [9, 11]:

$$\left\| \frac{d}{dt} S(t) f_\lambda \right\|_\infty \leq 2 \frac{\|f_\lambda\|_\infty}{t} \leq 2 \frac{\|f\|_\infty}{t} \qquad \text{for any } t > 0. \tag{4.9}$$

This implies that $u(t) \in \overline{\mathrm{Dom}(\mathcal{A}_\infty)}$. Notice that by Step 1 and Theorem 3.4, we know that $J_{u(t)} \subseteq J_{u(s)}$ and the corresponding assertion for the continuity points.

Step 3. Let $f \in L^N(\Omega)$. Then we know [11, 25] that $u(t) \in L^\infty(\Omega)$ for any $t > 0$, and the result follows as a consequence of Step 2. □

The analogous statement of Theorem 4.1 holds when the domain is $\mathbb{R}^N$. For simplicity, let us give in that case a geometric condition on the level sets of $f \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ which implies that $f \in \mathrm{Dom}(\mathcal{A}_\infty)$. We say that the set $E \subseteq \mathbb{R}^N$ satisfies the internal (external) r-ball condition if for any $x \in \partial E$ there exists $y_x \in \mathbb{R}^N$ such that $B(y_x, r) \subseteq E$ (resp., $B(y_x, r) \subseteq \mathbb{R}^N \setminus E$) modulo a Lebesgue null set and $x \in \partial B(y_x, r)$.

Proposition 4.8. *Let $f \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$. If $\{f > t\}$ satisfies the internal and external r-ball condition for almost any $t \geq 0$, then $f \in \mathrm{Dom}(\mathcal{A}_\infty)$ and there is $g \in \mathcal{A}_\infty f$ such that $\|g\|_\infty \leq \frac{N}{r}$.*

Proof. Let $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be the solution of

$$u - \lambda \,\mathrm{div}\left(\frac{Du}{|Du|} \right) = f \qquad \text{in } \mathbb{R}^N. \tag{4.10}$$

We know that $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\|u\|_\infty \leq \|f\|_\infty$. Using the notation introduced above, let us observe that a.e. in $x \in \mathbb{R}^N$ we have

$$h(x') := f^*(x) \chi_{B(y_x, r)}(x') - \|f\|_\infty (1 - \chi_{B(y_x,r)}(x')) \leq f(x'), \qquad x' \in \mathbb{R}^N \text{ a.e.}$$

Indeed, since f satisfies the internal r-ball condition, this follows easily from the definition of $f^*(x)$ given in section 2.

By the comparison principle, since the explicit solution of (4.10) with right-hand side h is given by the left-hand side of the next inequality (see, for instance, [11]), we have that

$$\max\left(-\|f\|_\infty, -\|f\|_\infty + \left(\|f\|_\infty + f^*(x) - \lambda\frac{N}{r}\right)\chi_{B(y,r)}(x')\right) \le u(x'), \quad x' \in \mathbb{R}^N \text{ a.e.}$$

Proceeding in an analogous way with the external ball condition, since

$$f(x') \le f_*(x)\chi_{B(y_x,r)}(x') + \|f\|_\infty(1 - \chi_{B(y_x,r)}(x')), \qquad x' \in \mathbb{R}^N \text{ a.e.}, \tag{4.11}$$

we obtain that

$$u(x') \le \max\left(\|f\|_\infty, \|f\|_\infty - \left(\|f\|_\infty - f_*(x) - \lambda\frac{N}{r}\right)\chi_{B(y_x,r)}(x')\right), \qquad x' \in \mathbb{R}^N \text{ a.e.} \tag{4.12}$$

From (4.11) and (4.12), we deduce that

$$\left\|\frac{u-f}{\lambda}\right\|_\infty \le \frac{N}{r}.$$

Since $\frac{u-f}{\lambda} \in \mathcal{A}_\infty u$, letting $\lambda \to 0+$ we deduce that $f \in \mathrm{Dom}(\mathcal{A}_\infty)$ and there is $g \in \mathcal{A}_\infty f$ such that $\|g\|_\infty \le \frac{N}{r}$ (see [17, 24]). □

5. Extensions and remarks. In this section we discuss some extensions of the previous results.

5.1. Boundary conditions. Theorem 3.4 is purely local, in the sense that it also holds considering Dirichlet boundary conditions in the minimization problem, and hence, by localization in appropriate balls, considering any kind of boundary condition.

The results concerning the evolution problem also hold in the case of Dirichlet boundary conditions or in $\mathbb{R}^N$ [10, 13, 11].

5.2. Anisotropic total variation. Let ϕ be a norm on $\mathbb{R}^N$. Following [2, 3], we say that ϕ is *smooth* if $\phi \in C^\infty(\mathbb{R}^N \setminus \{0\})$, and we say that ϕ is *elliptic* if there exist two constants $0 < c \le C < +\infty$ such that

$$c\,\mathrm{Id} \le \nabla^2\left(\frac{\phi(x)^2}{2}\right) \le C\,\mathrm{Id} \qquad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Given a function $f \in L^2(\Omega)$ and $\lambda > 0$ we consider the anisotropic version of problem (3.1):

$$\min_{u\in BV(\Omega)} \int_\Omega \phi(Du) + \frac{1}{2\lambda}\int_\Omega (u-f)^2\,dx, \tag{5.1}$$

where the integrand has to be suitably understood on the jump set J_u [8, sect. 5]. See [26] for some explicit examples in this setting.

Then Proposition 3.1 holds for the solution u of (5.1), provided the perimeter in (3.2) is replaced with the anisotropic perimeter

$$P_\phi(E,\Omega) \;:=\; \int_\Omega \phi(D\chi_E) \;=\; \int_{\partial^* E} \phi(\nu_E(x))\,d\mathcal{H}^{N-1}(x),$$

where $\partial^* E = J_{\chi_E}$ is the jump set defined in section 2, and ν_E is the corresponding normal vector. The following result follows from standard regularity theory [2, 3].

Proposition 5.1. *Let ϕ be smooth and elliptic. Let $f \in L^p(\Omega)$, with $p > N$, and let $u_\lambda \in BV(\Omega)$ be the (unique) minimizer of* (5.1)*. Then, for all $t \in \mathbb{R}$, the superlevel set $\{u_\lambda > t\}$ (resp., $\{u_\lambda \geq t\}$) has boundary of class $C^{1,\alpha}$, for all $\alpha < (p-N)/p$, out of a closed singular set Σ of Hausdorff dimension less than $N-2$.*

Reasoning as above, if $f \in BV(\Omega) \cap L^\infty(\Omega)$, we obtain that u_λ satisfies (3.3) also in the anisotropic setting. Moreover, the analogous statement as in Theorem 4.1 also holds, provided we substitute (4.1) with

$$\begin{aligned} &\frac{\partial u}{\partial t} = \operatorname{div}(\nabla\phi(Du)) && \text{in } Q_T =]0,T[\times \Omega, \\ &\nabla\phi(Du) \cdot \nu^\Omega = 0 && \text{in } Q_T =]0,T[\times \partial\Omega \end{aligned} \tag{5.2}$$

with an initial condition $f \in L^N(\Omega)$. Indeed, this statement follows as a consequence of two basic ingredients, the regularizing effect of (5.2) due to the homogeneity of the operator in its right-hand side and the L^N to L^∞ regularizing effect of the solutions of $\operatorname{div}(\nabla\phi(Du)) = f$. The proofs of these facts can be done as in the total variation case [25, 11]. As in section 5.1, we notice that Neumann boundary conditions may be replaced by Dirichlet ones and that we can also work in $\mathbb{R}^N$.

Remark 4. Notice that Theorems 3.4 and 4.1 cannot be expected to hold without further assumptions on the norm ϕ. Indeed, letting $N = 2$ and $\phi(x_1, x_2) = |x_1| + |x_2|$, from an example discussed in [16] it follows that we can find a set $E \subset \mathbb{R}^2$ (which is the union of two rectangles) such that, letting $f = \chi_E$, both the solution u_λ of (5.1) and the solution u of (5.2) have a jump set which strictly contains the jump set of f.

5.3. Convex functionals with linear growth. Let us now show that Theorems 3.4 and 4.1 also hold if we substitute (5.1) with a more general convex functional of the type

$$\int_\Omega F(Du) + \frac{1}{2\lambda}\int_\Omega (u-f)^2\,dx, \tag{5.3}$$

where $F(\xi) = \phi(\xi, -1)$, and $\phi : \mathbb{R}^{N+1} \to \mathbb{R}$ is a smooth and elliptic norm on $\mathbb{R}^{N+1}$. An important example is the Lagrangian $F(\xi) = \sqrt{1+|\xi|^2}$ of the minimal surface problem. Given a function $u \in L^p(\Omega)$, with $p \in [1, +\infty]$, we define $\widetilde{u} \in L^p(\Omega \times]0,1[)$ as $\widetilde{u}(x, x_{N+1}) = u(x) - x_{N+1}$. If $u \in BV(\Omega)$, then $\widetilde{u} \in BV(\Omega \times]0,1[)$, and using the coarea formula [7] it is easy to show that

$$\int_{\Omega\times]0,1[} \phi(D\tilde{u}) = \int_\Omega F(Du).$$

As a consequence, letting u_λ be the minimizer of (5.3), we have that $\widetilde{u}_\lambda$ is the unique minimizer of

$$\int_{\Omega\times]0,1[} \phi(Dv) + \frac{1}{2\lambda}\int_{\Omega\times]0,1[} (v-\widetilde{f})^2\,dx\,dx_{N+1},$$

among $v \in BV(\Omega \times]0,1[)$, with boundary conditions $v(x,0) = u(x)$ and $v(x,1) = u(x) - 1$, for $x \in \Omega$.

From the discussion above, if $f \in L^\infty(\Omega) \cap BV(\Omega)$, we get

$$J_{\widetilde{u}_\lambda} = J_{u_\lambda} \times]0,1[\subseteq J_{\widetilde{f}} = J_f \times]0,1[,$$

which yields, in particular, $J_{u_\lambda} \subseteq J_f$. Let us state the corresponding result for the evolution problem.

THEOREM 5.2. *Let $f \in L^\infty(\Omega)$. Let $u(t)$ be the solution of*

$$\begin{aligned} &\frac{\partial u}{\partial t} = \operatorname{div}(\nabla F(Du)) && \text{in } Q_T =]0,T[\times\Omega, \\ &\nabla F(Du)\cdot \nu^\Omega = 0 && \text{in } Q_T =]0,T[\times\partial\Omega \end{aligned} \tag{5.4}$$

with initial condition $u(0,x) = f(x)$. Then $u(t) \in L^\infty(\Omega)\cap BV(\Omega)$ for any $t > 0$, and

$$J_{u(t)} \subseteq J_{u(s)} \qquad \forall t > s > 0.$$

Moreover, if $u(s)$ is continuous at $x \in \Omega$, then so is $u(t)$ for any $t > s > 0$. If $f \in \overline{\operatorname{Dom}(\mathcal{A}_\infty)} \cap BV(\Omega)$, then the above assertions are true up to $s = 0$.

We have a corresponding statement for Dirichlet boundary conditions or for the Cauchy problem.

This result can be proved if we have a regularizing effect for the evolution problem, i.e., if as in the proof of Theorem 4.1 we are able to prove that

$$u(t) \in \overline{\operatorname{Dom}(-\operatorname{div}(\nabla F(Du)))} \quad \text{(where the closure is taken in } L^\infty(\Omega)).$$

This follows again from the estimate $\|u_t\|_\infty \le 2\frac{\|f\|_\infty}{t}$ which has been proved in [13] for the minimal surface operator (corresponding to $F(\xi) = \sqrt{1+|\xi|^2}$) and can be extended in a similar way to a general norm ϕ in $\mathbb{R}^{N+1}$.

Notice that we have restricted our statement to the case where $f \in L^\infty(\Omega)$, since we have no general L^N to L^∞ estimates for the equation $\operatorname{div}(\nabla F(Du)) = f$, without further assumptions on f or on the domain Ω.

5.4. Further remarks on the case $F(\xi) = \sqrt{1+|\xi|^2}$. To fix ideas, we shall work in $\mathbb{R}^N$. Let us consider the functional

$$\int_{\mathbb{R}^N} \sqrt{1+|Du|^2} + \frac{1}{2\lambda}\int_{\mathbb{R}^N} (u-f)^2\,dx, \tag{5.5}$$

which is used sometimes instead of functional (3.1) in problems related to image denoising and restoration. Our aim is to show that if f is discontinuous in some boundary, then, for small values of λ, the discontinuities are still preserved in the solution u_λ of (5.5). Moreover, the graph of u_λ has a vertical contact angle at the discontinuity.

Let us recall the following lemma, whose proof can be found in [6].

LEMMA 5.3. *Let $R, c > 0$. Then for any $\lambda^{-1} > \max\left(\frac{4N^2}{c}, \frac{2N}{cR}\right)$ there is a value of $\tilde{R} \in (0,R)$ such that there exists a radial solution $w_{\tilde{B}}$ of*

$$\begin{cases} w - \lambda \operatorname{div}\left(\dfrac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) = c & \text{in} \quad \tilde{B} = B(0,\tilde{R}), \\ w = 0 & \text{on} \quad \partial\tilde{B} \end{cases} \tag{5.6}$$

such that

$$0 > w'_{\tilde{B}}(r) > -\infty, \quad U < w_{\tilde{B}}(r) < c \quad \text{for } 0 < r < \tilde{R}, \text{ and}$$

$$w'_{\tilde{B}}(r) \to -\infty, \quad w_{\tilde{B}}(r) \to U \quad \text{as } r \to \tilde{R}^-$$

for some $U > 0$.

LEMMA 5.4. *For any $c > 0$ there is $\lambda_0 > 0$ such that for any $0 < \lambda \le \lambda_0$ there is $R_\lambda > 0$ such that the solution w_λ of (5.6) in $B(0, R_\lambda)$ satisfies $\inf_{\partial B(0,R_\lambda)} w_\lambda > 0$. Moreover, $w_\lambda \to c$ uniformly as $\lambda \to 0$.*

Proof. Let us choose $\lambda = 1$, $R = 1$, and $c' > 4N^2$ in Lemma 5.3. Let $\tilde{w}$ be the solution of (5.6) with right-hand side c' in a ball $\tilde{B}$ of radius $0 < \tilde{R} < 1$ given by that lemma. Let $g(x) = c' - \tilde{w}$ in $\tilde{B}$. Then $g = \mathrm{div}(\nabla g/\sqrt{1 + |\nabla g|^2})$. Let $c > 0$ and $\lambda_0 > 0$ be such that $c'\sqrt{\lambda_0} < c$. Then for any $\lambda \in (0, \lambda_0)$, $w_\lambda(x) = c - \sqrt{\lambda} g(\frac{x}{\sqrt{\lambda}})$ is the solution of (5.6) in $B(0, R_\lambda)$ with $R_\lambda = \sqrt{\lambda}\tilde{R}$ and satisfies $\inf_{\partial B(0,R_\lambda)} w_\lambda > 0$. The last assertion follows from the continuity of $\tilde{w}$. ∎

PROPOSITION 5.5. *Let Ω be an open bounded domain whose boundary is of class $C^{1,1}$, and let $f \in L^\infty(\mathbb{R}^N)$, $f \ge 0$, with $f \ge c > 0$ in Ω and $f = 0$ in $\mathbb{R}^N \setminus \Omega$. Let u_λ be the solution of*

$$
\left\{ \; u - \lambda \,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = f \qquad \text{in} \qquad \mathbb{R}^N. \right. \tag{5.7}
$$

Then for λ small enough u_λ is discontinuous on $\partial\Omega$, having a vertical contact angle.

We recall that if $u \in BV(\mathbb{R}^N)$ is a solution of (5.7) for some $f \in L^1(\mathbb{R}^N)$, then the vector field $Tu = \nabla u/\sqrt{1 + |\nabla u|^2}$ is such that $u - \lambda \,\mathrm{div}\, Tu = f$ in $\mathcal{D}'(\mathbb{R}^N)$ and $(Tu \cdot Du) = |\nabla u|^2/\sqrt{1 + |\nabla u|^2} + |D^s u|$.

Proof. Let us take $R > 0$ such that for any point $p \in \partial\Omega$ there are open balls B, B' of radius R such that $B \subseteq \Omega$, $B' \subseteq \mathbb{R}^N \setminus \overline{\Omega}$ and $p \in \partial B$, $p \in \partial B'$. Observe that, by the maximum principle (see [11]), we know that $u_\lambda \in L^2(\mathbb{R}^N)$ and $0 \le u_\lambda \le \|f\|_\infty$. First, we observe that u_λ is a supersolution of (5.6) on any ball $\tilde{B} \subseteq B$. By the comparison principle for (5.6) we obtain that $u_\lambda \ge u_{\tilde{B}} \ge U$ for some $U > 0$. Since we can do this for any ball $\tilde{B}$ inside Ω, we deduce that $u_\lambda \ge U$. Notice that, by Lemma 5.4, we may take λ and the balls $\tilde{B}$ small enough so that u_λ is greater than $\frac{c}{2}$ in $\partial\Omega$. On the other hand, u_λ is a subsolution of

$$
\left\{
\begin{array}{lll}
u - \lambda \,\mathrm{div}\left(\dfrac{Du}{\sqrt{1 + |Du|^2}} \right) = 0 & \text{in} & \tilde{B}', \\[2ex]
u = \|f\|_\infty & \text{on} & \partial \tilde{B}'
\end{array}
\right. \tag{5.8}
$$

for any ball $\tilde{B}'$ of radius smaller than R contained in $\mathbb{R}^N \setminus \overline{\Omega}$. Again, by Lemma 5.4, we know that for $\lambda > 0$ small enough, the solution u_λ is strictly below $\frac{c}{2}$ in $\partial\Omega$. We deduce that u_λ is discontinuous on $\partial\Omega$.

Let $Tu_\lambda = \nabla u_\lambda/\sqrt{1 + |\nabla u_\lambda|^2}$. Let us prove that u_λ has vertical contact angle from both sides of the discontinuity, i.e., $[Tu_\lambda \cdot \nu^\Omega] = -1$ and $[Tu_\lambda \cdot \nu^{\mathbb{R}^N \setminus \Omega}] = 1$. For that, let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then

$$
\begin{aligned}
\int_{\mathbb{R}^N} \mathrm{div}\, Tu_\lambda \, \varphi \, dx &= -\int_{\mathbb{R}^N} Tu_\lambda \cdot \nabla\varphi = -\int_\Omega Tu_\lambda \cdot \nabla \varphi - \int_{\mathbb{R}^N \setminus \Omega} Tu_\lambda \cdot \nabla\varphi \\
&= \int_\Omega \mathrm{div}\, Tu_\lambda \, \varphi + \int_{\mathbb{R}^N \setminus \Omega} \mathrm{div}\, Tu_\lambda \, \varphi \\
&\quad - \int_{\partial\Omega} [Tu_\lambda \cdot \nu^\Omega] \varphi - \int_{\partial\Omega} [Tu_\lambda \cdot \nu^{\mathbb{R}^N \setminus \Omega}] \varphi.
\end{aligned}
$$

That is,

(5.9)
$$\operatorname{div} Tu_\lambda = \operatorname{div} Tu_\lambda\, \chi_\Omega + \operatorname{div} Tu_\lambda\, \chi_{\mathbb{R}^N\setminus\Omega} - [Tu_\lambda \cdot \nu^\Omega]\mathcal{H}^{N-1}|_{\partial\Omega} - [Tu_\lambda \cdot \nu^{\mathbb{R}^N\setminus\Omega}]\mathcal{H}^{N-1}|_{\partial\Omega}.$$

Hence

$$\begin{aligned}\langle \operatorname{div} Tu_\lambda, u_\lambda\rangle &= \int_\Omega \operatorname{div} Tu_\lambda\, u_\lambda + \int_{\mathbb{R}^N\setminus\Omega} \operatorname{div} Tu_\lambda\, u_\lambda \\ &\quad - \int_{\partial\Omega} [Tu_\lambda \cdot \nu^\Omega] u_\lambda^* - \int_{\partial\Omega} [Tu_\lambda \cdot \nu^{\mathbb{R}^N\setminus\Omega}] u_\lambda^*,\end{aligned}$$

where $u_\lambda^* = \frac{u_\lambda^+ + u_\lambda^-}{2}$. Now

$$\begin{aligned}\int_{\mathbb{R}^N} \operatorname{div} Tu_\lambda\, u_\lambda\, dx &= -\int_{\mathbb{R}^N} Tu_\lambda \cdot Du_\lambda = -\int_\Omega Tu_\lambda \cdot Du_\lambda - \int_{\mathbb{R}^N\setminus\Omega} Tu_\lambda \cdot Du_\lambda \\ &\quad - \int_{\mathbb{R}^N} (Tu_\lambda \cdot Du_\lambda)^s\, d\mathcal{H}^{N-1}|_{\partial\Omega} \\ &= \int_\Omega \operatorname{div} Tu_\lambda\, u_\lambda + \int_{\mathbb{R}^N\setminus\Omega} \operatorname{div} Tu_\lambda\, u_\lambda - \int_{\mathbb{R}^N} (Tu_\lambda \cdot Du_\lambda)^s\, d\mathcal{H}^{N-1}|_{\partial\Omega} \\ &\quad - \int_{\partial\Omega} [Tu_\lambda \cdot \nu^\Omega] u_\lambda - \int_{\partial\Omega} [Tu_\lambda \cdot \nu^{\mathbb{R}^N\setminus\Omega}] u_\lambda.\end{aligned}$$

Comparing the above two expressions and using

$$(Tu_\lambda \cdot Du_\lambda)^s \mathcal{H}^{N-1}|_{\partial\Omega} = |(Du_\lambda)^s| \mathcal{H}^{N-1}|_{\partial\Omega} = |[u_\lambda]| \mathcal{H}^{N-1}|_{\partial\Omega}$$

(where $[u_\lambda]$ denotes the jump of u_λ on $\partial\Omega$), we deduce that

$$|[u_\lambda]| = \Big([Tu_\lambda \cdot \nu^\Omega] - [Tu_\lambda \cdot \nu^{\mathbb{R}^N\setminus\Omega}]\Big) \frac{[u_\lambda]}{2}.$$

Since $[u_\lambda] \neq 0$, this implies that

$$[Tu_\lambda \cdot \nu^{\mathbb{R}^N\setminus\Omega}] - [Tu_\lambda \cdot \nu^\Omega] = 2,$$

which in turn implies

$$[Tu_\lambda \cdot \nu^\Omega] = -1 \quad \text{and} \quad [Tu_\lambda \cdot \nu^{\mathbb{R}^N\setminus\Omega}] = 1,$$

since both $|[Tu_\lambda \cdot \nu^\Omega]|, |[Tu_\lambda \cdot \nu^{\mathbb{R}^N\setminus\Omega}]| \leq 1$. ∎

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